Etf 2024.godine - detaljno rešeni zadaci

1) $a^3-27 = a^3-3^3 = (a-3)(a^2+3a+9)$

$a^2-2a-3=0 \rightarrow a_{1,2} = \dfrac{2\pm 4}{2}$ $\nearrow a_1 = 3$, $\searrow a_2=-1$

$a^2-2a-3=(a-3)(a+1)$

$(a+3)^2-3a = a^2+6a+9-3a = a^2+3a+9$

вратимо се на задатак

$$\frac{\cancel{(a-3)}\cancel{(a^2+3a+9)}}{\cancel{(a-3)^2}} \cdot \frac{\cancel{(a-3)}(a+1)}{\cancel{a^2+3a+9}} = a+1$$

2) $(1+i)^{2024} = ((1+i)^2)^{1012} = (2i)^{1012} = 2^{1012}\cdot \underbrace{i^{1012}}_{=1} = 2^{1012}$

$(1-i)^{2024} = ((1-i)^2)^{1012} = (-2i)^{1012} = 2^{1012}\cdot i^{1012} = 2^{1012}$

$(1+i)^{2025}\cdot i = (1+i)\cdot i\cdot (1+i)^{2024} = (i+i^2)\cdot 2^{1012}$
$= (-1+i)\cdot 2^{1012}$

$(1-i)^{2025}\cdot i = (1-i)\cdot i\cdot (1-i)^{2024} = (i-i^2)\cdot 2^{1012}$
$= (1+i)\cdot 2^{2012}$

ЗАМЕНИМО!

$$\frac{2^{1012}+(1+i)2^{1012}}{2^{1012}+(-1+i)\cdot 2^{1012}} = \frac{\cancel{2^{1012}}(1+1+i)}{\cancel{2^{1012}}(1-1+i)}$$

$$= \frac{2+i}{i}\cdot\frac{(-i)}{(-i)} = \frac{-2i-i^2}{-i^2} = \frac{-2i+1}{1} = 1-2i$$

3) $49^{1-\log_7 2} + 5^{-\log_5 4} =$

$\dfrac{49^1}{49^{\log_7 2}} + \dfrac{1}{5^{\log_5 4}} =$ $\qquad A^{\log_A B} = B$

користимо

$\dfrac{49}{7^{2\log_7 2}} + \dfrac{1}{4} =$ $\qquad \log_A B^n = n\log_A B$

$\dfrac{49}{7^{\log_7 4}} + \dfrac{1}{4} = \dfrac{49}{4} + \dfrac{1}{4} = \dfrac{50}{4} = \dfrac{25}{2}$

4. $x^2 + (m-2)x + m = 0$ $\qquad x_1^2 + x_2^2 \le 8 - 9x_1x_2$

$a = 1 \quad b = m-2 \quad c = m$

$x_1 + x_2 = -\frac{b}{a} = -(m-2) = -m+2$

$x_1 \cdot x_2 = \frac{c}{a} = \frac{m}{1} = m$

$(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 \rightarrow x_1^2 + x_2^2 = (x_1+x_2)^2 - 2x_1x_2$

$(x_1+x_2)^2 - 2x_1x_2 \le 8 - 9x_1x_2$

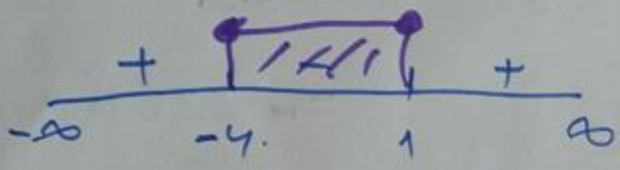
$(x_1+x_2)^2 + 7x_1x_2 - 8 \le 0$

$(2-m)^2 + 7m - 8 \le 0$

$4 - 4m + m^2 + 7m - 8 \le 0$

$m^2 + 3m - 4 \le 0$

$m_{1,2} = \frac{-3 \pm 5}{2} \quad \begin{matrix} \nearrow m_1 = 1 \\ \searrow m_2 = -4 \end{matrix}$



$m \in [-4, 1]$

разлика $m = 1$ и $m = -4$ је

$1 - (-4) = 1 + 4 = 5$

5. $\left(2^{x}-\frac{3}{4^{x}}\right)^{9}$ упоређујемо са $(a+b)^{n}$

$a=2^{x}$

$b=-\frac{3}{4^{x}}=-3\cdot 2^{-2x}$

$n=9$

$$\begin{aligned}T_{k+1}&=\binom{n}{k}a^{n-k}b^{k}\\&=\binom{9}{k}\left(2^{x}\right)^{9-k}\cdot\left(-3\cdot 2^{-2x}\right)^{k}\\&=\binom{9}{k}2^{x(9-k)}\cdot(-3)^{k}\cdot 2^{-2xk}\\&=\binom{9}{k}(-3)^{k}\cdot 2^{\underbrace{9x-xk-2xk}}\end{aligned}$$

ово упоређујемо са 2^{0}

$9x-xk-2xk=0$

$9x-3xk=0$

$3x(3-k)=0 \Rightarrow x=0 \lor k=3$

за $k=3$

$\Rightarrow T_{4}=\binom{9}{3}(-3)^{3}=\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\cdot(-27)$

$=-2268$

6.

$$x^4 - 3x^2 + 2 = 0 \qquad x^2 = t$$

$$t^2 - 3t + 2 = 0 \rightarrow t_{1,2} = \frac{3 \pm 1}{2} \begin{array}{l} \nearrow t_1 = 2 \\ \searrow t_2 = 1 \end{array}$$

$$\Downarrow$$

$$(t-2)(t-1) = 0$$

$$(x^2-2)(x^2-1) = 0$$

$$x_1 = \sqrt{2} \qquad x_3 = 1$$

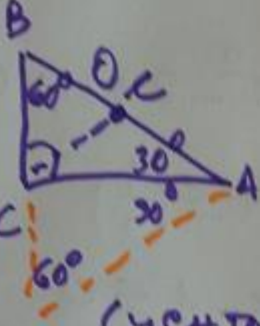
$$x_2 = -\sqrt{2} \qquad x_4 = -1$$

$$x^4 - 3x^2 + 2 = (x-\sqrt{2})(x+\sqrt{2})(x-1)(x+1)$$

сад правимо $x^2 + px + q$, значи можемо да помножимо било кад

$C_2^4 = \binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$ 1ба

7. Гледамо само базу



$BC = \frac{c}{2}$

$AC = \left(\text{висина } h = \frac{a\sqrt{3}}{2}\right) = \frac{c\sqrt{3}}{2}$

$AO = BO = \frac{c}{2} = CO = SO$

(центар описане кружнице је на средини хипотенузе (висина H)

$V = \frac{1}{3} B \cdot H = \frac{1}{3} \frac{a \cdot b}{2} \cdot H$

$V = \frac{1}{6} \frac{c}{2} \cdot \frac{c\sqrt{3}}{2} \cdot \frac{c}{2} = \frac{c^3\sqrt{3}}{48}$

сад погледамо и ово нема понуђено, па мало преправимо:

$\frac{c^3\sqrt{3}}{48} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{c^3 \cdot 3}{48\sqrt{3}} = \frac{c^3}{16\sqrt{3}}$

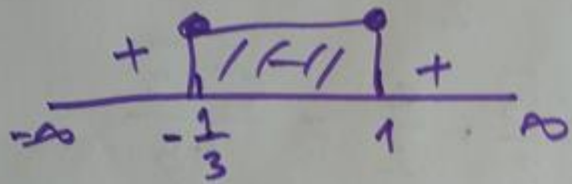
8. $y=\sqrt{\log_{\frac{1}{2}}(3x^2-2x)}$

због корена: $\log_{\frac{1}{2}}(3x^2-2x)\geq 0$

$$3x^2-2x\leq\left(\frac{1}{2}\right)^0$$

$$3x^2-2x-1\leq 0$$

$$x_{1,2}=\frac{2\pm 4}{6}\quad x_1=1,\quad x_2=-\frac{1}{3}$$



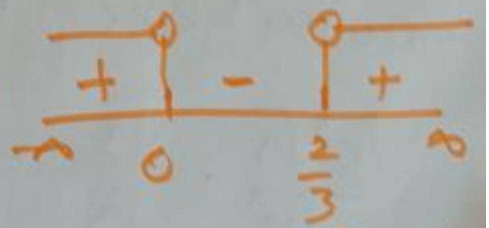
$x\in\left[-\frac{1}{3},1\right]$

због логаритма:

$$3x^2-2x>0$$

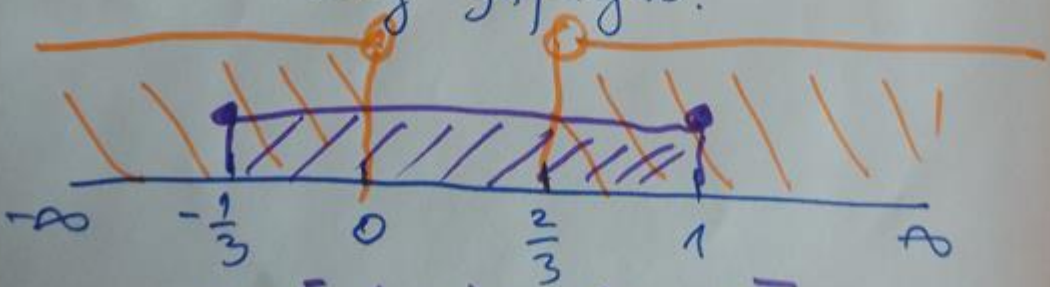
$$x(3x-2)>0$$

$x_1=0\quad x_2=\frac{2}{3}$



$x\in(-\infty,0)\cup\left(\frac{2}{3},\infty\right)$

сад заједно!



$x\in\left[-\frac{1}{3},0\right)\cup\left(\frac{2}{3},1\right]$

9. $-\frac{3}{5} \le \underbrace{x + x^2 + \dots + x^n + \dots} < 1$

$S_n = \frac{a}{1-q}$ бесконачна геом. прогресија

$-\frac{3}{5} \le \frac{x}{1-x} < 1$ за $|x| < 1$

$-1 < x < 1$

$-\frac{3}{5} \le \frac{x}{1-x}$ и $-1 < x < 1$

$-3(1-x) \le 5x$

$-3 + 3x \le 5x$

$3x - 5x \le 3$

$-2x \le 3$

$x \geq -\frac{3}{2}$

$x \in (-1, 1)$

$\frac{x}{1-x} < 1 \quad / \cdot (1-x) > 0$ увек

$x < 1 - x$

$2x < 1$

$x < \frac{1}{2}$

$\Rightarrow x \in \left(-1, \frac{1}{2}\right)$

заедно:

$x \in \left(-1, \frac{1}{2}\right)$

10. $\sqrt[3]{x-3}+\sqrt[3]{x-1}=\sqrt[3]{2x-4} \quad /()^3$

$$x-3+3\cdot\sqrt[3]{(x-3)^2}\cdot\sqrt[3]{x-1}+3\cdot\sqrt[3]{x-3}\cdot\sqrt[3]{(x-1)^2}+x-1=2x-4$$

$$\cancel{2x-4}+3\cdot\sqrt[3]{x-3}\,\sqrt[3]{x-1}\cdot\left(\sqrt[3]{x-3}+\sqrt[3]{x-1}\right)=\cancel{2x-4}$$

$$3\cdot\sqrt[3]{x-3}\,\sqrt[3]{x-1}\cdot\left(\sqrt[3]{x-3}+\sqrt[3]{x-1}\right)=0$$

$\sqrt[3]{x-3}=0 \Rightarrow x=3$

$\sqrt[3]{x-1}=0 \Rightarrow x=1$

$\sqrt[3]{x-3}+\sqrt[3]{x-1}=0$

$\sqrt[3]{x-3}=-\sqrt[3]{x-1} \quad ()^3$

$x-3=-(x-1)$

$x-3=-x+1$

$2x=4$

$x=2$

$3+1+2=6$

11.

$$f_1(x)=\sqrt{(x-1)^2}=|x-1|=\begin{cases} x-1 & \text{за } x\geq 1 \\ -(x-1) & \text{за } x<1 \end{cases}$$

$$f_2(x)=\frac{x^2-1}{x+1}=\frac{(x-1)(x+1)}{x+1}=x-1 \quad \text{за } x\neq -1$$

$$f_3(x)=x^2-\frac{x^3+1}{x+1}=\frac{\cancel{x^3}+x^2-\cancel{x^3}-1}{x+1}=\frac{x^2-1}{x+1}=x-1$$

за $x\neq-1$

$$f_4(x)=2^{\log_2(x-1)}=x-1 \quad \text{за } x-1>0$$

$x>1$

према условима закључујемо $f_2=f_3$

12. $x^2+(y-25)^2=225$

$p=0 \quad q=25 \quad r^2=225$

То је угао између тангенти $(0,0)$

$t: \ y=kx+n \qquad \underset{x \ \ y}{(0,0)}$

$0=0+n$

$\Rightarrow n=0$

услов додира $\quad r^2(k^2+1)=(kp-q+n)^2$

$225(k^2+1)=(0-25+0)^2$

$225(k^2+1)=625$

$k^2+1=\frac{625}{225}$

$k^2=\frac{625}{225}-1$

$k^2=\frac{400}{225}=\frac{16}{9}$

$k_1=-\frac{4}{3} \quad k_2=\frac{4}{3}$

$tg\alpha=\left|\frac{k_2-k_1}{1+k_1k_2}\right|$

$tg\alpha=\left|\frac{\frac{4}{3}+\frac{4}{3}}{1-\frac{4}{3}\cdot\frac{4}{3}}\right|$

$tg\alpha=\left|\frac{\frac{8}{3}}{1-\frac{16}{9}}\right|=\left|\frac{\frac{8}{3}}{-\frac{7}{9}}\right|$

$tg\alpha=\left|-\frac{24}{7}\right|=\frac{24}{7} \qquad \alpha=arctg\,\frac{24}{7}$

13. $x^2+y^2+2x \le 1$ → унутрашњост кружнице

$x-y+a=0$ ⇒ $y=x+a$

права да буде њена тангента

$x^2+(x+a)^2+2x \le 1$

$x^2+x^2+2ax+a^2+2x-1 \le 0$ $\quad D=0$ за квадрат

$2x^2+\underbrace{(2a+2)}x+\underbrace{a^2-1} \le 0$

$A=2 \quad B=2a+2 \quad C=a^2-1$

$$D = B^2-4AC = (2a+2)^2-4\cdot 2\cdot(a^2-1)$$
$$= 4a^2+8a+4-8a^2+8$$
$$= -4a^2+8a+12 = 0$$
$$a^2-2a-3=0$$
$$a_{1,2} = \frac{2\pm 4}{2} \quad \begin{matrix} \nearrow a_1 = 3 \\ \searrow a_2 = -1 \end{matrix}$$

а може и Виет:

$$a_1\cdot a_2 = \frac{c}{a} = -\frac{3}{1} = -3$$

140

$\alpha:\beta:\gamma = 6:7:11$

$\alpha + \beta + \gamma = 360$

$6k + 7k + 11k = 360$

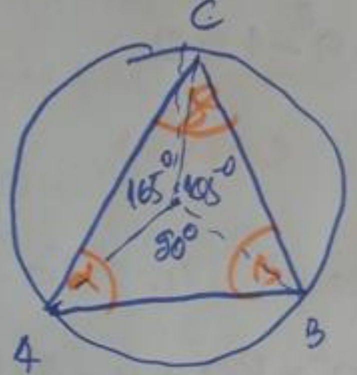
$24k = 360$

$k = 15$

$\alpha = 6k = 6 \cdot 15 = 90^\circ$

$\beta = 7k = 7 \cdot 15 = 105^\circ$

$\gamma = 11k = 7 \cdot 15 = 165^\circ$



$\alpha = 2\beta$

над истим луком!

углови Δ су:

$\alpha = \frac{105^\circ}{2} = \cancel{57{,}75^\circ}\ 57{,}5^\circ$

$\beta = \frac{165^\circ}{2} = 82{,}5^\circ$

$\gamma = \frac{90^\circ}{2} = 45^\circ$

15. $\sin x - \sqrt{3}\cos x \neq 0$ услов

$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x \neq 0$ $[0, \pi]$

$\sin x \cdot \cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3} \neq 0$

$\sin\left(x - \frac{\pi}{3}\right) \neq 0 \rightarrow$ $x - \frac{\pi}{3} \neq k\pi$

$x \neq \frac{\pi}{3} + k\pi$

за $k=0 \rightarrow x \neq \frac{\pi}{3}$

$\underbrace{\cos 5x + \cos 3x}_{\text{формула}} + \cos 4x = 0$

$2\cos\frac{5x+3x}{2}\cos\frac{5x-3x}{2} + \cos 4x = 0$

$2\cos 4x \cos x + \cos 4x = 0$

$\cos 4x\,(2\cos x + 1) = 0$

$\cos 4x = 0$ $2\cos x + 1 = 0$

$4x = \frac{\pi}{2} + k\pi \rightarrow x = \frac{\pi}{8} + \frac{k\pi}{4}$ $\cos x = -\frac{1}{2}$

за $k=0 \rightarrow x = \frac{\pi}{8}$ $x = \frac{2\pi}{3} + 2k\pi \lor x = \frac{4\pi}{3} + 2k\pi$

за $k=1 \rightarrow x = \frac{3\pi}{8}$ за $k=0 \rightarrow x = \frac{2\pi}{3}$ нема реш.

за $k=2 \rightarrow x = \frac{5\pi}{8}$

за $k=3 \rightarrow x = \frac{7\pi}{8}$

16.

$$\lim_{x\to 0} \frac{e^{\sin^2 x} - \cos x}{\sqrt{x^2+1} - 1} = \frac{e^0 - 1}{\sqrt{1} - 1} = \frac{0}{0} \overset{L.T.}{=}$$

$$\lim_{x\to 0} \frac{e^{\sin^2 x}\cdot(\sin^2 x)' + \sin x}{\frac{1}{2\sqrt{x^2+1}}\cdot(x^2+1)' - 0} =$$

$$\lim_{x\to 0} \frac{e^{\sin^2 x}\cdot 2\sin x\cdot\cos x + \sin x}{\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}} =$$

$$\lim_{x\to 0} \frac{\overbrace{\sin x}^{\to 1}\left(2e^{\sin^2 x}\cdot\cos x + 1\right)\cdot\sqrt{x^2+1}}{x}$$

ПОЗНАТИ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$= \left(2e^0\cdot 1 + 1\right)\cdot\sqrt{0^2+1} = 3\cdot 1 = 3$$

17.

$P = 15\sqrt{3}\ cm^2$

$R = \frac{14\sqrt{3}}{3}\ cm$

$a = 10\ cm$

$b + c = ?$

$P = \frac{a \cdot b \cdot c}{4R}$

$10 \cdot b \cdot c = 15\cancel{\sqrt{3}} \cdot 4 \cdot \frac{14\cancel{\sqrt{3}}}{\cancel{3}}$

$10 \cdot b \cdot c = 840$

$\boxed{b \cdot c = 84}$

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$

$\frac{10}{\sin\alpha} = 2 \cdot \frac{14\sqrt{3}}{3} \rightarrow \sin\alpha = \frac{30}{28\sqrt{3}} = \frac{15}{14\sqrt{3}}$

$\sin\alpha = \frac{5\sqrt{3}}{14}$

$\sin^2\alpha + \cos^2\alpha = 1$

$\cos^2\alpha = 1 - \frac{75}{196} = \frac{196-75}{196} = \frac{121}{196}$

$\boxed{\cos\alpha = +\frac{11}{14}}$ (3δοι ουμ/ποι γωνία)

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$100 = b^2 + c^2 - 2 \cdot \cancel{84}^{6} \cdot \frac{11}{\cancel{14}}$

$100 = b^2 + c^2 - 132$

$\boxed{b^2 + c^2 = 232}$

$(b+c)^2 = b^2 + 2bc + c^2$

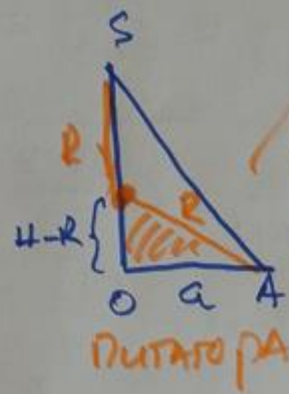
$(b+c)^2 = 232 + 2 \cdot 84$

$(b+c)^2 = 232 + 168$

$(b+c)^2 = 400$

$\boxed{b + c = 20}$

18.



$$(H-R)^2+a^2=R^2$$

$$H^2-2RH+\cancel{R^2}+a^2=\cancel{R^2}$$

$$H^2-2RH+a^2=0$$

$$H(H-2R)+a^2=0$$

$$a^2=2RH-H^2$$

$$V=\frac{1}{3}\,B\cdot H$$

$$V=\frac{1}{3}\,\frac{6a^2\sqrt{3}}{4}\cdot H$$

$$V=\frac{a^2\sqrt{3}\cdot H}{2} \longrightarrow V=\frac{(2RH-H^2)\sqrt{3}\cdot H}{2}$$

$$V=\frac{(2RH^2-H^3)\sqrt{3}}{2}$$

$$V'_H=\frac{\sqrt{3}}{2}\left(4RH-3H^2\right)=0$$

$$4RH-3H^2=0$$

$$H(4R-3H)=0$$

$$4R=3H$$

$$H=\frac{4R}{3}$$

$$a^2=2\cdot R\cdot\frac{4R}{3}-\left(\frac{4R}{3}\right)^2$$

$$a^2=\frac{8R^2}{3}-\frac{16R^2}{9}=\frac{8R^2}{9}$$

$$V=\frac{\sqrt{3}}{\cancel{2}}\,\frac{8R^2}{9}\cdot\frac{\cancel{4}^{\,2}R}{3}$$

$$V=\frac{16\sqrt{3}\,R^3}{27}$$

19. услов $9-3^{x^2-3x+2}>0$

$9>3^{x^2-3x+2}$

$3^2>3^{x^2-3x+2}$

$2>x^2-3x+2 \Rightarrow x^2-3x<0$

$+$	0	$-$	3	$+$

$x\in(0,3)$

$5^{2x}-6\cdot 5^{x+1}+5^3=0$

$5^{2x}-6\cdot 5^x\cdot 5+5^3=0$ $\quad 5^x=t$

$t^2-30t+125=0 \to t_1=5,\ t_2=25$

$(x^2-4)(5^{2x}-6\cdot 5^{x+1}+5^3)\le 0$

$(x-2)(x+2)(5^x-5)(5^x-25)\le 0$

$5^x-5>0$ $\quad 5^x-5<0$

$5^x>5$ $\quad 5^x<5$

$x>1$ $\quad x<1$

$5^x-25>0$ $\quad 5^x-25<0$

$5^x>25$ $\quad x<2$

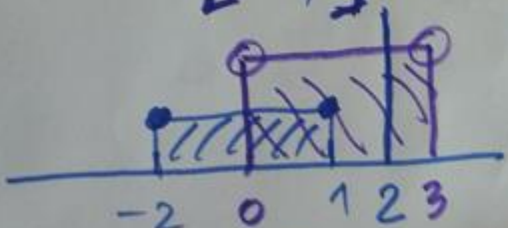
$x>2$

	-2		1		2	
$x-2$	$-$		$-$	$-$		$+$
$x+2$	$-$		$+$	$+$		$+$
5^x-5	$-$		$-$	$+$		$+$
5^x-25	$-$		$-$	$-$		$+$
Р	$+$		$-$	$+$		$+$

може и 2 јер $0\le 0$

$x\in[-2,1]\cup\{2\}$

$x\in[-2,1]$ → сад ово пресечемо са условом



$x\in(0,1]\cup\{2\}$

20. $1, 2, 3, 4, 5$ $\quad 6, 7, 8, 9$

фиксирајмо 4

$\boxed{4}\ \boxed{\geq 5 \neq 8}\ \boxed{\leq 5}\ \boxed{\leq 5} \rightarrow 1 \cdot C_1^3 \cdot C_2^4 = 18$

а није 4

$\boxed{4}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8}\ \boxed{\leq 5} \rightarrow 1 \cdot C_2^3 \cdot C_1^4 = 12$

а није 4

$\boxed{4}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8} \rightarrow 1 \cdot C_3^3 \rightarrow 1$

укупно 31

фиксирајмо 8

$\boxed{8}\ \boxed{\leq 5 \neq 4}\ \boxed{\leq 5 \neq 4}\ \boxed{\leq 5 \neq 4} \rightarrow 1 \cdot C_3^4 = 4$

$\boxed{8}\ \boxed{> 5 \neq 8}\ \boxed{\leq 5 \neq 4}\ \boxed{\leq 5 \neq 4} \rightarrow 1 \cdot C_1^3 \cdot C_2^4 = 18$

$\boxed{8}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8}\ \boxed{\leq 5 \neq 4} \rightarrow 1 \cdot C_2^3 \cdot C_1^4 = 12$

$\boxed{8}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8} \rightarrow 1 \cdot C_3^3 = 1$

укупно 35

фиксирајмо 4 и 8

$\boxed{4}\ \boxed{8}\ \boxed{\leq 5 \neq 4}\ \boxed{\leq 5 \neq 4} \rightarrow 1 \cdot 1 \cdot C_2^4 = 6$

$\boxed{4}\ \boxed{8}\ \boxed{\leq 5 \neq 4}\ \boxed{\geq 5 \neq 8} \rightarrow 1 \cdot 1 \cdot C_1^4 \cdot C_1^3 = 12$

$\boxed{4}\ \boxed{8}\ \boxed{\geq 5 \neq 8}\ \boxed{\geq 5 \neq 8} \rightarrow 1 \cdot 1 \cdot C_2^3 = 3$

укупно 21

$31 + 35 + 21 = 87$