Primer 4.

Funkciju \( f(x) = |x| - 1 \) razviti u Furijeov red na segmentu \([-1, 1]\) a zatim izračunati sumu reda \( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \).

Rešenje:

Kako je \( f(-x) = |-x| - 1 = |x| - 1 = f(x) \) zaključujemo da je funkcija parna.

Koristimo formule:

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)
\]

\[
a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx
\]

\[
a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \quad b_n = 0
\]

\[
a_0 = \frac{1}{l} \int_{-1}^{1} f(x) dx = \frac{1}{l} \int_{-1}^{1} (x-1) dx = 2 \int_{0}^{1} (x-1) dx = 2 \left( \frac{x^2}{2} - x \right) \bigg|_{0}^{1} = -1
\]

\[
a_n = \frac{1}{l} \int_{-1}^{1} f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{l} \int_{-1}^{1} (x-1) \cos \frac{n\pi x}{l} dx = 2 \int_{0}^{1} (x-1) \cos \frac{n\pi x}{l} dx
\]

Kao i uvek, ovaj integral ćemo rešiti na stranu uz pomoć parcijalne integracije:

\[
\int (x-1) \cos \frac{n\pi x}{l} dx = \left. \frac{x-1 = u}{du} \cos \frac{n\pi x}{l} dx = dv \right| = (x-1) \cdot \frac{1}{n\pi} \sin \frac{n\pi x}{l} \cdot \int \frac{1}{n\pi} \sin \frac{n\pi x}{l} dx =
\]

\[
= \frac{(x-1) \sin \frac{n\pi x}{l}}{n\pi} - \frac{1}{n\pi} \int \sin \frac{n\pi x}{l} dx = \frac{(x-1) \sin \frac{n\pi x}{l}}{n\pi} + \frac{1}{n\pi} \frac{1}{n\pi} \cos \frac{n\pi x}{l}
\]

\[
= \frac{(x-1) \sin \frac{n\pi x}{l}}{n\pi} + \frac{1}{(n\pi)^2} \cos \frac{n\pi x}{l}
\]

Sad se vratimo da ubacimo granice:

\[
a_n = 2 \int_{0}^{1} (x-1) \cos \frac{n\pi x}{l} dx = 2 \left( \frac{(x-1) \sin \frac{n\pi x}{l}}{n\pi} + \frac{1}{(n\pi)^2} \cos \frac{n\pi x}{l} \right) \bigg|_{0}^{1} =
\]

\[
= 2 \left[ \left( \frac{(1-1) \sin \frac{n\pi 1}{n\pi}}{n\pi} + \frac{1}{(n\pi)^2} \cos \frac{n\pi 1}{n\pi} \right) - \left( (0-1) \sin \frac{n\pi 0}{n\pi} + \frac{1}{(n\pi)^2} \cos \frac{n\pi 0}{n\pi} \right) \right]
\]

\[
= 2 \left[ \frac{1}{(n\pi)^2} \cos n\pi - \frac{1}{(n\pi)^2} \right] = \frac{2}{(n\pi)^2} (\cos n\pi - 1) = \frac{2}{(n\pi)^2} ((-1)^n - 1)
\]
Slično kao u prethodnim primerima, razmišljamo o parnim i neparnim \( n \), pa je:

\[
a_n = \begin{cases} 
-\frac{4}{(n\pi)^2}, & n = 2k - 1 \\
0, & n = 2k 
\end{cases}
\]

Sad idemo u početnu formulu:

\[
f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)
\]

\[
f(x) = |x| - 1 = \frac{1}{2}(-1) + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos \left( \frac{(2k-1)\pi x}{l} \right)
\]

\[
|x| - 1 = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2k-1)\pi x}{(2k-1)^2}
\]

Pogledajmo i sumu koja se traži: \( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \). Vidimo da u našem redu treba ubaciti \( x = 0 \):

\[
|0| - 1 = -\frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}
\]

\[
-1 = -\frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}
\]

\[
\frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{2} \rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}
\]

**Primer 5.**

Funkciju \( f(x) = x - 2 \) razviti u Furijeov red na segmentu \([1,3]\).

**Rešenje:**

Moramo koristiti formule:

\[
f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right)
\]

\[
a_0 = \frac{2}{b-a} \int_{a}^{b} f(x) dx \quad a_n = \frac{2}{b-a} \int_{a}^{b} f(x) \cos \frac{2n\pi x}{b-a} dx \quad b_n = \frac{2}{b-a} \int_{a}^{b} f(x) \sin \frac{2n\pi x}{b-a} dx
\]

Dakle, imamo:
\[ a_0 = \frac{2}{3-1} \int_1^3 (x-2) \, dx = \left( \frac{x^2}{2} - 2x \right)^{\frac{3}{1}} = \left( \frac{3^2}{2} - 6 \right) - \left( \frac{1^2}{2} - 2 \right) = 0 \]

\[ a_n = \frac{2}{3-1} \int_1^3 (x-2) \cos \frac{2n \pi x}{3-1} \, dx = \]
\[ = \int_1^3 (x-2) \cos n \pi x \, dx \]

Da rešimo najpre ovo bez granica:

\[ \int_1^3 (x-2) \cos n \pi x \, dx = \left| \begin{array}{l} x-2 = u \\ dx = du \end{array} \right| 1 \sin n \pi x = v \]
\[ = (x-2) \cdot \frac{1}{n \pi} \sin n \pi x - \int \frac{1}{n \pi} \sin n \pi x \, dx = \]
\[ = \frac{(x-2) \sin n \pi x}{n \pi} - \frac{1}{n \pi} \int \sin n \pi x \, dx = \frac{(x-2) \sin n \pi x}{n \pi} + \frac{1}{n \pi} \cos n \pi x \]
\[ = \frac{(x-2) \sin n \pi x}{n \pi} + \frac{1}{(n \pi)^2} \cos n \pi x \]

\[ a_n = \int_1^3 (x-2) \cos n \pi x \, dx = \left( \frac{(x-2) \sin n \pi x}{n \pi} + \frac{1}{(n \pi)^2} \cos n \pi x \right)^{\frac{3}{1}} = \]
\[ = \left( \frac{(3-2) \sin n \pi 3}{n \pi} + \frac{1}{(n \pi)^2} \cos n \pi 3 \right) - \left( \frac{(1-2) \sin n \pi 1}{n \pi} + \frac{1}{(n \pi)^2} \cos n \pi 1 \right) = \]
\[ = \frac{\sin 3n \pi}{n \pi} + \frac{1}{(n \pi)^2} \cos 3n \pi + \frac{\sin n \pi}{n \pi} - \frac{1}{(n \pi)^2} \cos n \pi \]
\[ = \frac{1}{(n \pi)^2} \cos 3n \pi - \frac{1}{(n \pi)^2} \cos n \pi \]
\[ = \frac{1}{(n \pi)^2} \cos 3n \pi - \frac{1}{(n \pi)^2} \cos n \pi \]

Sećate se trigonometrijske formulice: \[ \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \], ako nju upotrebimo:
\[ a_n = \frac{1}{(n \pi)^2} \left[ \cos 3n \pi - \cos n \pi \right] = \frac{1}{(n \pi)^2} \left[ -2 \sin 2n \pi \cdot \sin n \pi \right] = 0 \rightarrow a_n = 0 \]

Još da nadjemo:

\[ b_n = \frac{2}{3-1} \int_1^3 (x-2) \sin \frac{2n \pi x}{3-1} \, dx = \int_1^3 (x-2) \sin n \pi x \, dx \]

3
\[ \int (x-2) \sin n \pi x \, dx = \left. \frac{x-2 = u}{dx = du} \sin n \pi x \, dv = v \right| = -(x-2) \cdot \frac{1}{n \pi} \cos n \pi x + \int \frac{1}{n \pi} \cos n \pi x \, dx = \]

\[ = -\frac{(x-2) \cos n \pi x}{n \pi} + \frac{1}{n \pi} \int \cos n \pi x \, dx = -\frac{(x-2) \cos n \pi x}{n \pi} + \frac{1}{n \pi} \frac{1}{n \pi} \sin n \pi x \]

\[ = -\frac{(x-2) \cos n \pi x}{n \pi} + \frac{1}{(n \pi)^2} \sin n \pi x \]

Da ubacimo granice:

\[ b_n = \int_1^{\pi} (x-2) \sin n \pi x \, dx = \left( \frac{-(x-2) \cos n \pi x}{n \pi} + \frac{1}{(n \pi)^2} \sin n \pi x \right) \bigg|_1^{\pi} = \]

\[ \left( \frac{-(3-2) \cos n \pi 3}{n \pi} + \frac{1}{(n \pi)^2} \sin n \pi 3 \right) - \left( \frac{-(1-2) \cos n \pi x}{n \pi} + \frac{1}{(n \pi)^2} \sin n \pi 1 \right) = \]

\[ = \frac{-\cos n \pi 3}{n \pi} - \frac{\cos n \pi x}{n \pi} = -\frac{1}{n \pi} \left( \cos 3n \pi + \cos n \pi \right) \]

Opet mora formulica: \[ \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \]

\[ b_n = -\frac{1}{n \pi} \left( \cos 3n \pi + \cos n \pi \right) = -\frac{1}{n \pi} \left[ \cos \frac{2n \pi x}{b-a} \right] \cos n \pi = -\frac{2}{n \pi} (-1)^n = \frac{2}{n \pi} (-1)^{n+1} \]

\[ b_n = (-1)^{n+1} \frac{2}{n \pi} \]

Sad idemo u početnu formulu:

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n \pi x}{b-a} + b_n \sin \frac{2n \pi x}{b-a} \right) \]

Pazite:

\[ f(1-0)=1, \ f(1+0)=1 \quad \text{i} \quad f(3-0)=1, \ f(3+0)=-1 \]

pogledajte sliku:
Funkciju $f(x) = \begin{cases} x, & x \in (0,1) \\ 2-x, & x \in [1,2] \end{cases}$ razviti u red po:

a) po sinusima

b) po cosinusima

Rešenje:

a) Funkciju $f(x) = \begin{cases} x, & x \in (0,1) \\ 2-x, & x \in [1,2] \end{cases}$ razviti u red po sinusima.

Da bi smo razvili ovu funkciju po sinusima, moramo je dodefinisati do neparne funkcije.

To ćemo obaviti na sledeći način:

$$F(x) = \begin{cases} 2-x, & x \in [1,2] \\ x, & x \in (-1,1) \\ -2-x, & x \in [-2,-1] \end{cases}$$
Pogledajmo kako ova funkcija izgleda na slici:

\[
\begin{align*}
&\text{Naravno da su ovde } a_0 \text{ i } a_n \text{ jednaki nuli a tražimo: } b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \\
&b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx = \frac{2}{2l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{2} \, dx = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{2} \, dx \\
&\text{Zbog načina na koji je funkcija definisana, ovaj integral rastavljamo na dva:} \\
&b_n = \frac{2}{2l} \int_{0}^{l} x \sin \frac{n\pi x}{2} \, dx + \frac{2}{l} \int_{0}^{l} (2-x) \sin \frac{n\pi x}{2} \, dx \\
&\text{Nakon rešavanja ovih integrala, metodom parcijalne integracije, na sličan način kao u prethodnim primerima dobijamo:} \\
&b_n = \frac{8}{n^2 \pi^2} \sin \frac{n\pi}{2} \\
&\text{Razmišljamo kako se ponaša } \sin \frac{n\pi}{2}. \text{ Znamo da } n \text{ uzima vrednosti } 1, 2, 3... \\
\text{Za } n = 1 \quad \sin \frac{n\pi}{2} = \sin \frac{\pi}{2} = 1 \\
\text{Za } n = 2 \quad \sin \frac{n\pi}{2} = \sin \frac{2\pi}{2} = 0 \\
\text{Za } n = 3 \quad \sin \frac{n\pi}{2} = \sin \frac{3\pi}{2} = -1 \\
\text{Za } n = 4 \quad \sin \frac{n\pi}{2} = \sin \frac{4\pi}{2} = 0 \\
\text{Za } n = 5 \quad \sin \frac{n\pi}{2} = \sin \frac{5\pi}{2} = 1 \\
\text{Za } n = 6 \quad \sin \frac{n\pi}{2} = \sin \frac{6\pi}{2} = 0 \\
\text{itd.} 
\end{align*}
\]
Dakle, zaključujemo: \( b_n = \begin{cases} (-1)^k \frac{8}{(2k+1)^3 \pi^2}, & n = 2k \ 0, & n = 2k + 1 \end{cases} \quad k=0,1,2,3 \ldots \)

Pa je:

\[
f(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \sin \frac{(2k+1)\pi x}{2}, za x \in (0,2]
\]

\[
F(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \sin \frac{(2k+1)\pi x}{2}, za x \in [-2,2]
\]

b)

Za razvoj po kosinusima moramo dodefinisati funkciju do parne na sledeći način:

\[
F(x) = \begin{cases} x + 2, & x \in [-2,-1] \\ |x|, & x \in (-1,1) \\ x - 2, & x \in [1,2] \end{cases}
\]

Data funkcija je prikazana na sledećoj slici:

![Graph of the function](image)

Naravno, sada je \( b_n = 0 \) a tražimo: \( a_0 = \frac{1}{l} \int_{-l}^{l} f(x)dx \quad a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \)

\[
a_0 = \frac{1}{l} \int_{-l}^{l} f(x)dx = \frac{2}{l} \int_{0}^{l} f(x)dx
\]

\[
a_0 = \frac{2}{2} \int_{0}^{2} f(x)dx = \int_{0}^{2} f(x)dx = \int_{0}^{1} xdx + \int_{1}^{2} (2 - x)dx = 1
\]
\[ a_n = \frac{1}{l} \int f(x) \cos \frac{n \pi x}{l} \, dx = \frac{2}{l} \int f(x) \cos n \frac{\pi x}{l} \, dx \]
\[ a_n = \frac{2}{2} \int f(x) \cos \frac{n \pi x}{2} \, dx = \frac{2}{2} \int f(x) \cos n \frac{\pi x}{2} \, dx = \int x \cos n \frac{\pi x}{2} \, dx + \int (2-x) \cos n \frac{\pi x}{2} \, dx \]

Parcijalnom integracijom rešimo ove integrale i dobijamo:

\[ a_n = \frac{8}{n^2 \pi^2} \cos \frac{n \pi}{2} - \frac{4}{n^2 \pi^2} (1 + \cos n \pi) \]

Razmislimo kako se ponaša izraz \( \cos \frac{n \pi}{2} \) za različite \( n \).

za \( n=1 \) \( \cos \frac{\pi}{2} = \cos \frac{\pi}{2} = 0 \)

za \( n=2 \) \( \cos \frac{2 \pi}{2} = \cos \frac{2 \pi}{2} = -1 \)

za \( n=3 \) \( \cos \frac{3 \pi}{2} = \cos \frac{3 \pi}{2} = 0 \)

za \( n=4 \) \( \cos \frac{4 \pi}{2} = \cos \frac{4 \pi}{2} = 1 \)

\( \ldots \)

Dakle, ako je \( n \) neparan broj, \( n=2k+1 \), tada je \( a_n = 0 \)

Pogledajmo sada parne \( n \), ali oblika \( n=4k \) ili \( n=4k+2 \) za \( k=0,1,2,3 \ldots \)

\( n=4k \)

\[ a_n = \frac{8}{n^2 \pi^2} \cos \frac{n \pi}{2} - \frac{4}{n^2 \pi^2} (1 + \cos n \pi) \]
\[ a_{4k} = \frac{8}{(4k)^2 \pi^2} \cos \frac{4k \pi}{2} - \frac{4}{(4k)^2 \pi^2} (1 + \cos 4k \pi) = \frac{8}{16k^2 \pi^2} \cos 2k \pi - \frac{4}{16k^2 \pi^2} (1+1) \]
\[ = \frac{8}{16k^2 \pi^2} \cos 2k \pi - \frac{8}{16k^2 \pi^2} \cos 2k \pi = 0 \]
\[ n = 4k + 2 \]

\[ a_n = \frac{8}{n^2 \pi^2} \cos \frac{n \pi}{2} - \frac{4}{n^2 \pi^2} (1 + \cos n \pi) \]

\[ a_{2k} = \frac{8}{(4k + 2)^2 \pi^2} \cos \frac{(4k + 2) \pi}{2} - \frac{4}{(4k + 2)^2 \pi^2} (1 + \cos (4k + 2) \pi) \]

\[ = \frac{\mathcal{A}}{(2k + 1)^2 \pi^2} \cos \frac{(2k + 1) \pi}{2} - \frac{\mathcal{A}}{(2k + 1)^2 \pi^2} (1 + \cos 2 \pi (2k + 1)) \]

\[ = -\frac{2}{(2k + 1)^2 \pi^2} \cos (2k + 1) \pi - \frac{1}{(2k + 1)^2 \pi^2} (1 + 1) \]

\[ = -\frac{2}{(2k + 1)^2 \pi^2} = -\frac{2}{(2k + 1)^2 \pi^2} \]

Konačno imamo:

\[ f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos (2k + 1) \pi x}{(2k + 1)^2}, \quad x \in (0, 2) \]

\[ F(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos (2k + 1) \pi x}{(2k + 1)^2}, \quad x \in [-2, 2] \]