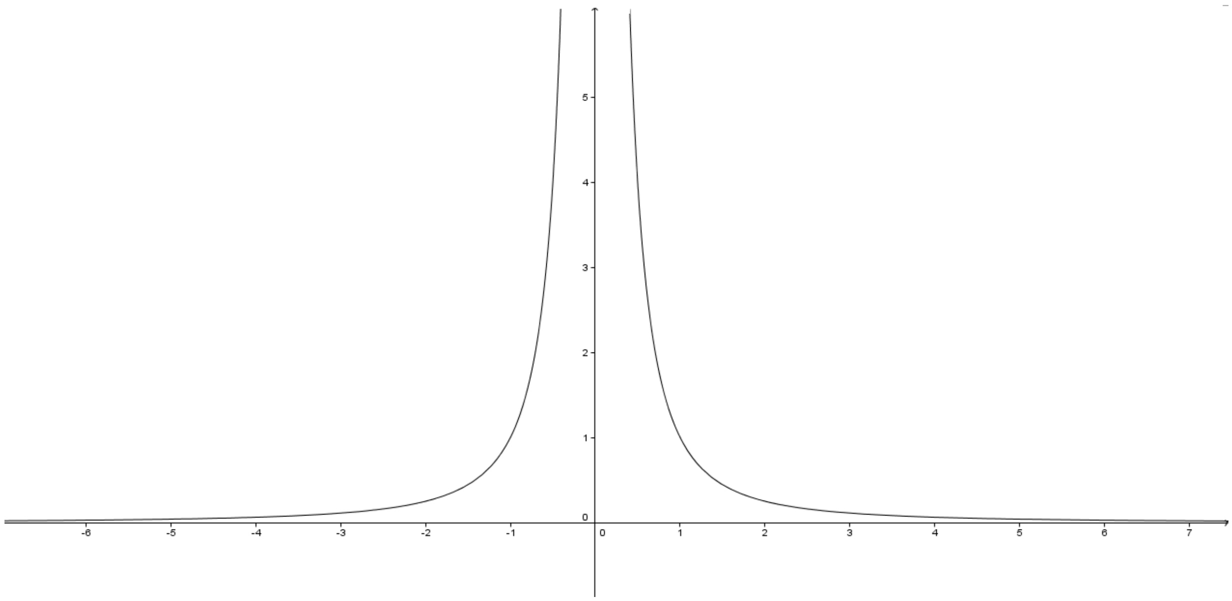
Nesvojstveni integrali – zadaci

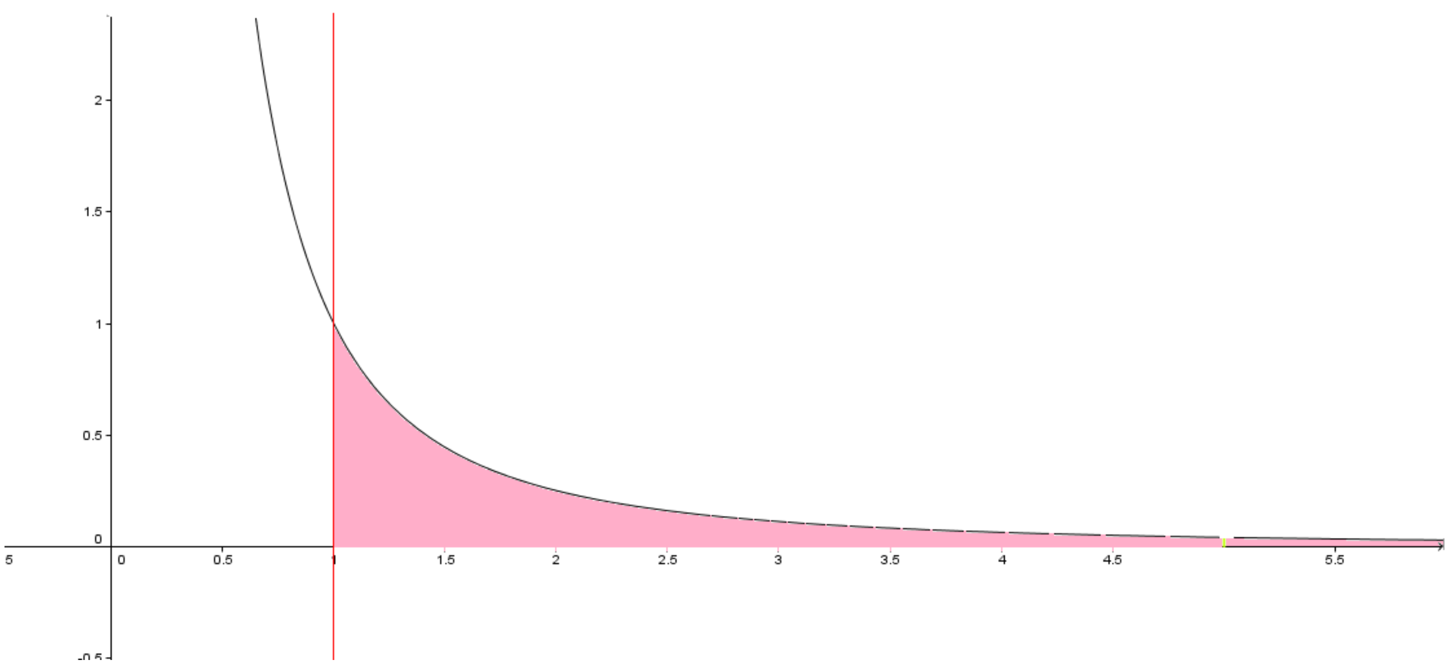
Primer 1.

$$\int_1^{\infty} \frac{1}{x^2} dx = ?$$

Podsetimo se najpre kako izgleda ova kriva $y = \frac{1}{x^2}$



Nama treba od 1 do ∞ pa bi to bilo:



Dakle, rešenje ovog integrala daje označenu površinu. Da li je ona beskonačna?

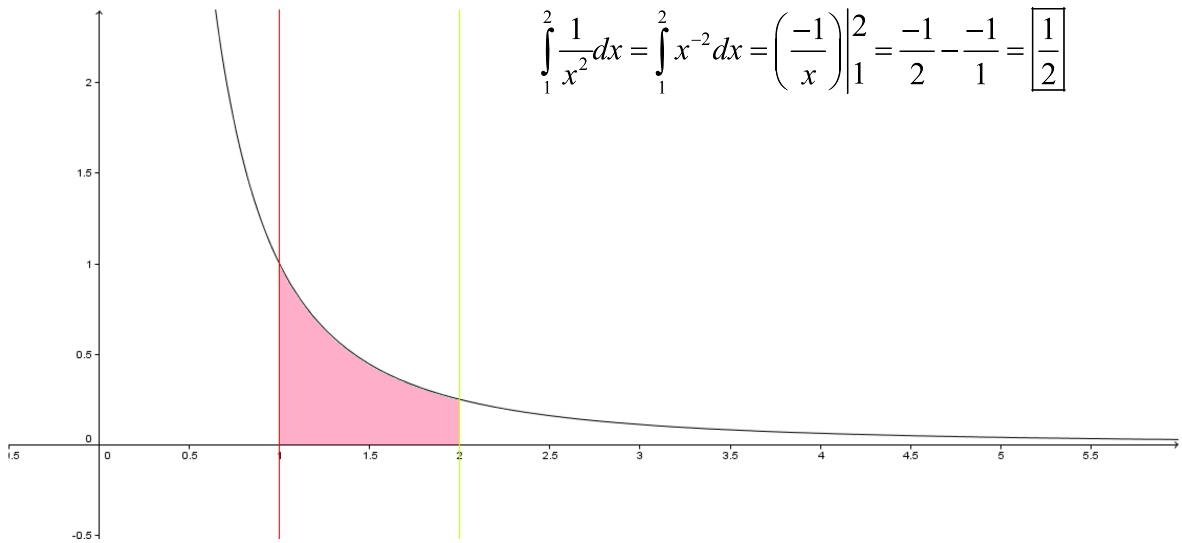
Kako je teoretski $\int_a^{+\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$, imamo:

$$\int_1^{\infty}\frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b x^{-2}dx = \lim_{b\to\infty}\left(\frac{x^{-2+1}}{-2+1}\right)\Bigg|_1^b = \lim_{b\to\infty}\left(\frac{-1}{x}\right)\Bigg|_1^b =$$
$$\lim_{b\to\infty}\left(\frac{-1}{b}-\frac{-1}{1}\right) = -\frac{1}{\infty}+1 = 0+1 = 1$$

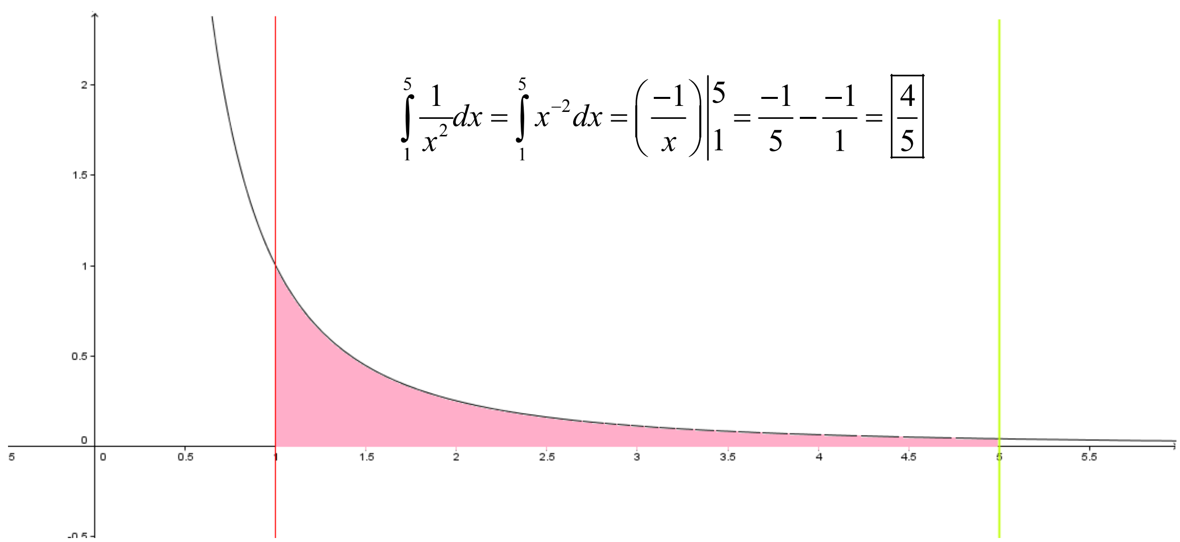
Površina ispod krive $y=\frac{1}{x^2}$ u granicama od 1 do ∞ je jednaka **jedan**.

Ovo nam deluje malo čudno, pa pogledajmo kolike su površine za recimo :

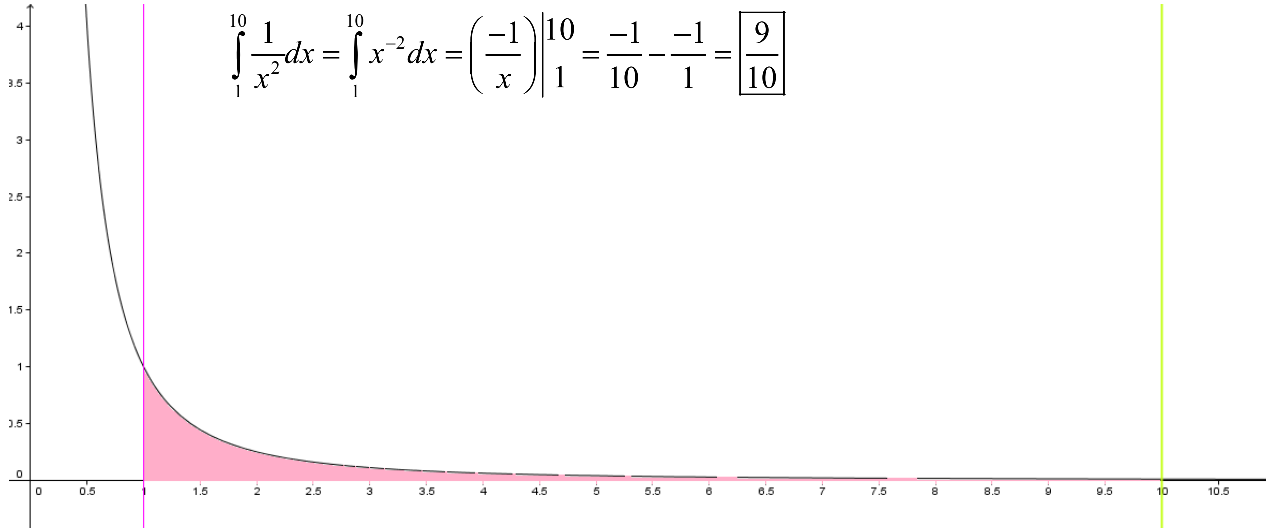
$$\int_1^2 \frac{1}{x^2}dx = \int_1^2 x^{-2}dx = \left(\frac{-1}{x}\right)\Bigg|_1^2 = \frac{-1}{2}-\frac{-1}{1} = \boxed{\frac{1}{2}}$$



$$\int_1^5 \frac{1}{x^2}dx = \int_1^5 x^{-2}dx = \left(\frac{-1}{x}\right)\Bigg|_1^5 = \frac{-1}{5}-\frac{-1}{1} = \boxed{\frac{4}{5}}$$



$$\int_1^{10} \frac{1}{x^2}dx = \int_1^{10} x^{-2}dx = \left(\frac{-1}{x}\right)\Big|_1^{10} = \frac{-1}{10} - \frac{-1}{1} = \boxed{\frac{9}{10}}$$

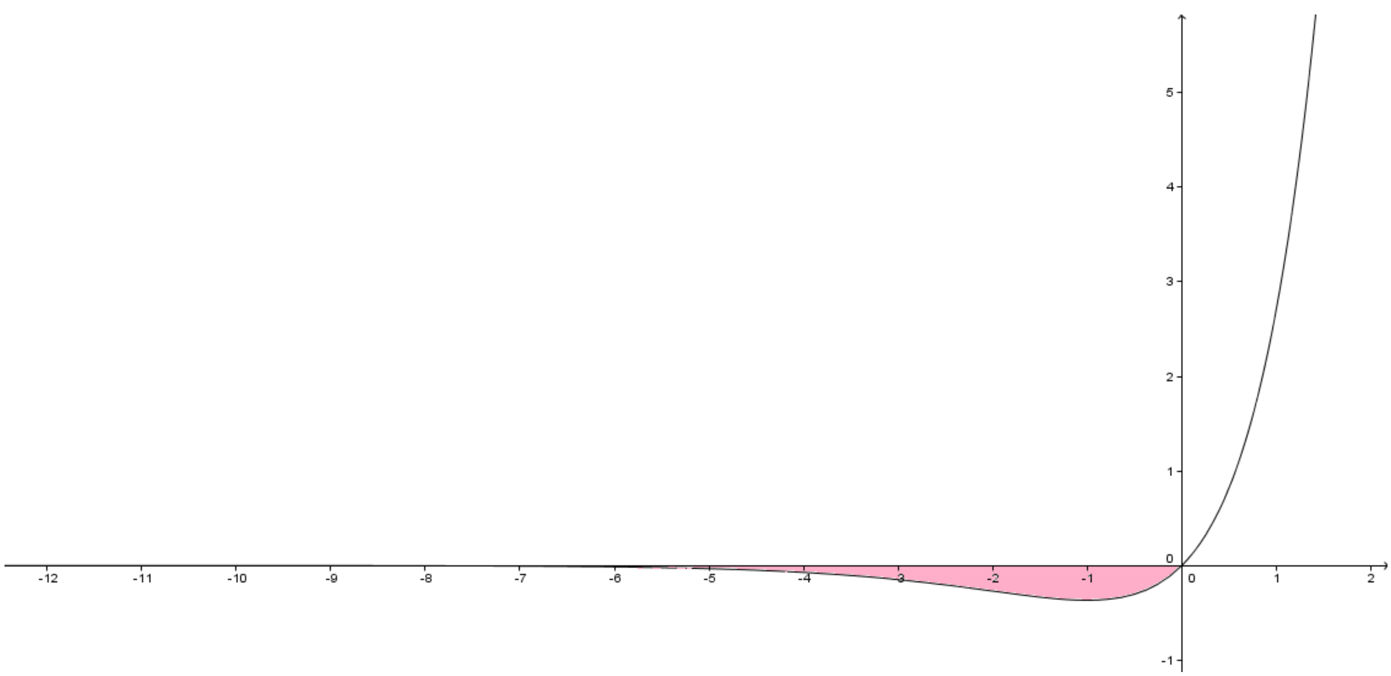


Vidimo da se površina pri povećanju gornje granice povećava i da se za približava jedinici.

Primer 2.

$$\int_{-\infty}^{0} xe^x dx = ?$$

Da najpre pogledamo šta bi to značilo na skici:



Dakle, to bi bila ova osenčena površina.

Kako je teoretski $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$, za naš integral imamo:

$\int_{-\infty}^{0} xe^x dx = \lim_{a\to-\infty} \int_{a}^{0} xe^x dx$, rešićemo najpre integral bez granica, parcijalnom integracijom:

$$\int xe^x dx = \left| \begin{array}{ll} x = u & e^x dx = dv \\ dx = du & \int e^x dx = v \\ & e^x = v \end{array} \right| == x \cdot e^x - \int e^x dx = xe^x - e^x = \boxed{e^x(x-1)}$$

Vratimo se na zadatak:

$$\int_{-\infty}^{0} xe^x dx = \lim_{a\to-\infty} \int_{a}^{0} xe^x dx = \lim_{a\to-\infty} e^x(x-1)\Big|_{a}^{0} = \lim_{a\to-\infty}[e^0(0-1) - e^a(a-1)] =$$

$$= -1 - \lim_{a\to-\infty} e^a(a-1)$$

Sad imamo:

$$\lim_{a\to-\infty} e^a(a-1) = \lim_{a\to-\infty} \frac{(a-1)}{e^{-a}} = \frac{-\infty}{\infty} = lopital = \lim_{a\to-\infty} \frac{1}{-e^{-a}} = \frac{1}{-\infty} = 0$$

Dakle:

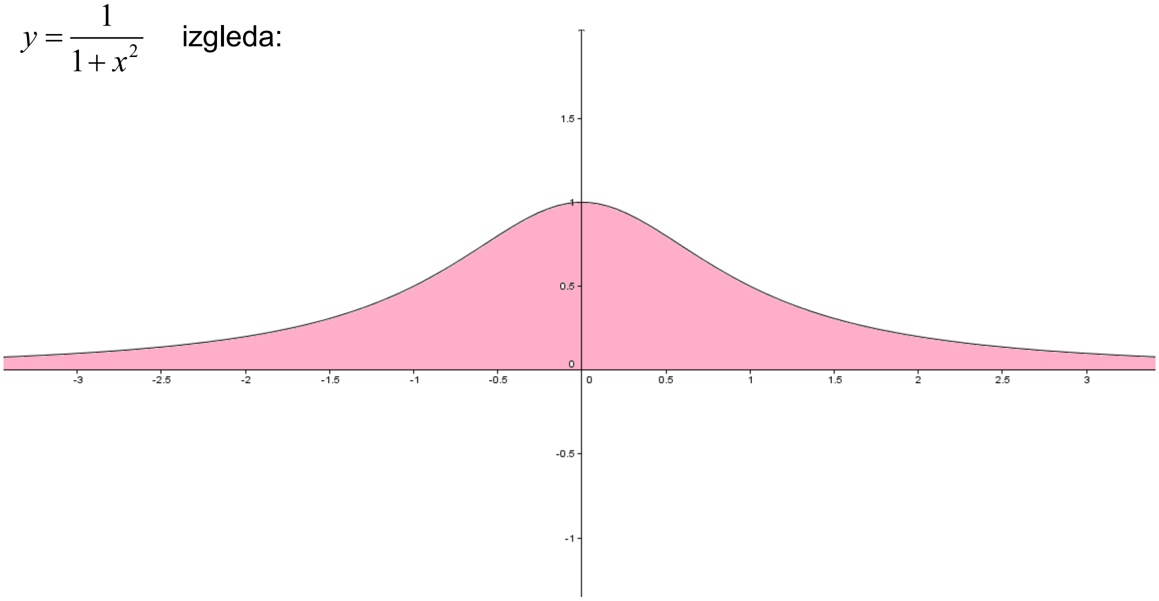
$$\boxed{\int_{-\infty}^{0} xe^x dx = -1}$$

Naravno, površina bi bila apsolutna vrednost (jer je kriva ispod x-ose).

Primer 3.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = ?$$

Kriva $y = \dfrac{1}{1+x^2}$ izgleda:



Rešavanjem datog integrala dobijamo površinu osenčenu na slici.

Postoje dva načina za rešavanje, mi ćemo vam pokazati oba, a vi radite kako kaže vaš profesor .

I način

Teoretski imamo $\int\limits_{-\infty}^{\infty} f(x)dx = \lim\limits_{\substack{b\to\infty \\ a\to-\infty}} \int\limits_{a}^{b} f(x)dx$, pa je

$$\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim\limits_{\substack{b\to\infty \\ a\to-\infty}} \int\limits_{a}^{b} \frac{1}{1+x^2}dx = \lim\limits_{\substack{b\to\infty \\ a\to-\infty}} (arctgx)\Big|_{a}^{b} =$$

$$\lim\limits_{\substack{b\to\infty \\ a\to-\infty}} (arctgb - arctga) = arctg\infty - arctg(-\infty) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

II način

Teoretski imamo $\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{c} f(x)dx + \int\limits_{c}^{+\infty} f(x)dx$, a za našu situaciju je zgodno uzeti da je c = 0.

$$\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int\limits_{-\infty}^{0} \frac{1}{1+x^2}dx + \int\limits_{0}^{+\infty} \frac{1}{1+x^2}dx \to \text{Radimo svaki posebno}$$

$$\lim\limits_{a\to-\infty} \int\limits_{a}^{0} \frac{1}{1+x^2}dx = \lim\limits_{a\to-\infty} (arctgx)\Big|_{a}^{0} = \lim\limits_{a\to-\infty} (arctg0 - arctga) = -arctg(-\infty) = -(-\frac{\pi}{2}) = \frac{\pi}{2}$$

$$\lim\limits_{b\to\infty} \int\limits_{0}^{b} \frac{1}{1+x^2}dx = \lim\limits_{b\to\infty} (arctgx)\Big|_{0}^{b} = \lim\limits_{b\to\infty} (arctgb - arctg0) = arctg(\infty) = \frac{\pi}{2}$$

Sad imamo :

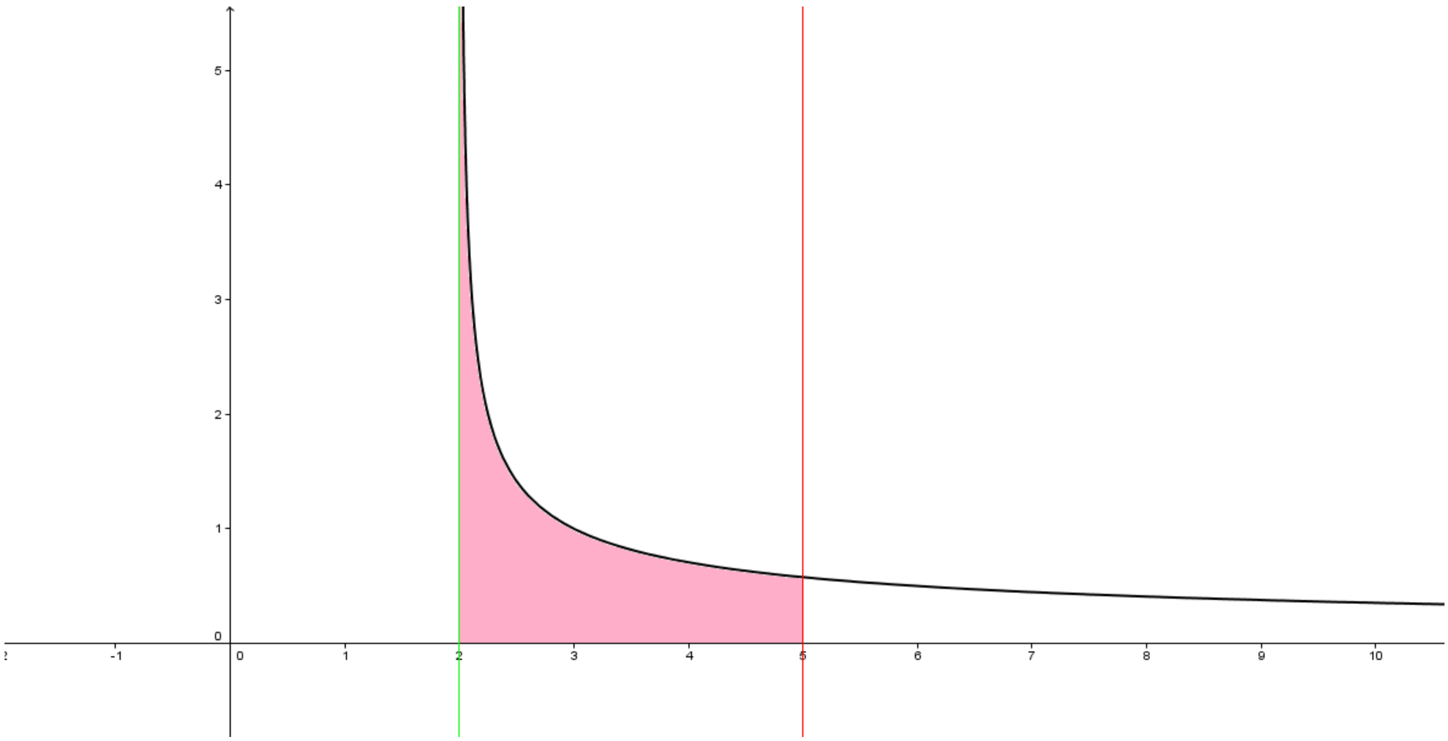
$$\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int\limits_{-\infty}^{0} \frac{1}{1+x^2}dx + \int\limits_{0}^{+\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Primer 4.

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = ?$$

Primetimo da je oblast definisanosti ove funkcije x > 2, a nama treba integral od 2 do 5.

Na skici to bi izgledalo:



Da se podsetimo šta kaže teorija:

Ako je f(x) neograničena u okolini tačke a (to jest prava x = a je vertikalna asimptota sleva) i neprekidna u svakom intervalu [a+ε , b] , ε >0 onda je :

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)dx$$

Imamo dakle:

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{\varepsilon \to 0+} \int_{2+\varepsilon}^5 \frac{1}{\sqrt{x-2}}dx$$

Rešimo bez granica dati integral $\int \frac{1}{\sqrt{x-2}}dx = \left| \begin{matrix} x-2=t^2 \\ dx = 2tdt \end{matrix} \right| = \int \frac{2tdt}{t} = 2t = 2\sqrt{x-2}$

$$\lim_{\varepsilon \to 0+} \int_{2+\varepsilon}^5 \frac{1}{\sqrt{x-2}}dx = \lim_{\varepsilon \to 0+} \left(2\sqrt{x-2}\right)\Big|_{2+\varepsilon}^{5} = \lim_{\varepsilon \to 0+} \left(2\sqrt{5-2} - 2\sqrt{2+\varepsilon-2}\right) = \lim_{\varepsilon \to 0+} \left(2\sqrt{3} - 2\sqrt{\varepsilon}\right) = 2\sqrt{3}$$

Primer 5.

Ispitati konvergenciju integrala $\int\limits_0^2 \frac{x^3}{\sqrt{4-x^2}}dx$

Naš posao je da rešimo ovaj integral, pa ako dobijemo konačnu vrednost- konvergira, a ko dobijemo beskonačno, onda divergira.

Oblast definisanosti podintegralne funkcije je:

$$\left.\begin{array}{l} 4-x^2 \geq 0 \\ 4-x^2 \neq 0 \end{array}\right\} \rightarrow 4-x^2 > 0 \rightarrow x \in (-2,2)$$

Sad teoretski koristimo :

Ako funkcija f(x) nije ograničena u nekoj okolini tačke b
(to jest prava x = b je vertikalna asimptota sdesna),
tada , ako je funkcija f(x) neprekidna na svakom intervalu [a,b-ε], ε >0 je

$$\int\limits_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int\limits_a^{b-\varepsilon} f(x)dx$$

$$\int\limits_0^2 \frac{x^3}{\sqrt{4-x^2}}dx = \lim_{\varepsilon\to 0+} \int\limits_0^{2-\varepsilon} \frac{x^3}{\sqrt{4-x^2}}dx$$

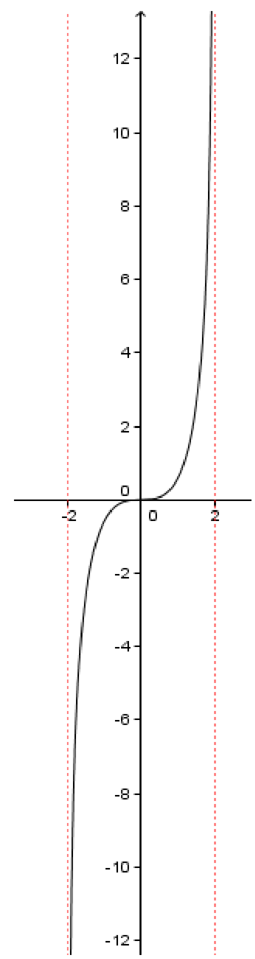
$$\int \frac{x^3}{\sqrt{4-x^2}}dx = \int \frac{x^2\cdot xdx}{\sqrt{4-x^2}} = \left| \begin{array}{l} 4-x^2=t^2 \\ -2xdx = 2tdt \\ xdx = -tdt \\ 4-x^2 = t^2 \rightarrow x^2 = 4-t^2 \end{array} \right| =$$

$$\int \frac{(4-t^2)(-tdt)}{t} = -\int (4-t^2)dt = \frac{t^3}{3} - 4t = \frac{\sqrt{(4-x^2)^3}}{3} - 4\sqrt{(4-x^2)}$$

$$\lim_{\varepsilon\to 0+} \int\limits_0^{2-\varepsilon} \frac{x^3}{\sqrt{4-x^2}}dx = \lim_{\varepsilon\to 0+} \left(\frac{\sqrt{(4-x^2)^3}}{3} - 4\sqrt{(4-x^2)} \right) \Bigg|_{0}^{2-\varepsilon} =$$

$$\boxed{\lim_{\varepsilon\to 0+} \underbrace{\left(\frac{\sqrt{(4-4+4\varepsilon-\varepsilon^2)^3}}{3} - 4\sqrt{(4-4+4\varepsilon-\varepsilon^2)} \right)}_{OVO\ JE\ 0}} - \frac{\sqrt{4^3}}{3} + 4\sqrt{4} =$$

$$-\frac{8}{3} + 8 = \frac{16}{3}$$



Zaključujemo da integral KONVERGIRA!

Primer 6.

$$\int_0^2 \frac{1}{\sqrt[5]{(x-1)^2}}dx = ?$$

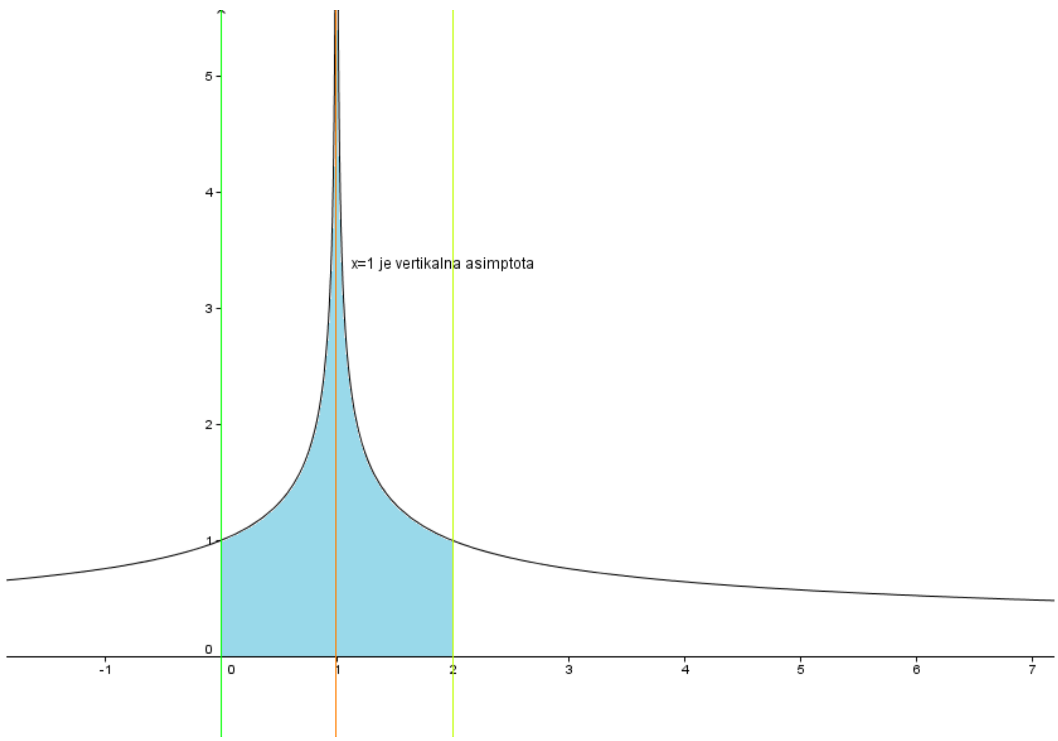
Naša podintegralna funkcija je definisana za $x-1 \neq 0 \rightarrow x \neq 1$.

Teoretski radimo:

Ako je situacija da je f(x) neograničena u okolini tačke c $\in$ (a , b)
(to jest prava x = c je vertikalna asimptota)

i ako je f(x) neprekidna u svakom intervalu [a,c - ε] , [c + ε , b] , ε >0 onda je :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^b f(x)dx$$



$$\int_0^2 \frac{1}{\sqrt[5]{(x-1)^2}}dx = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} (x-1)^{-\frac{2}{5}} dx + \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 (x-1)^{-\frac{2}{5}} dx =$$

Najpre je : $\int (x-1)^{-\frac{2}{5}} dx = \dfrac{(x-1)^{-\frac{2}{5}+1}}{-\frac{2}{5}+1} = \dfrac{(x-1)^{\frac{3}{5}}}{\frac{3}{5}}$

$$\lim_{\varepsilon \to 0+} \frac{5}{3}(x-1)^{\frac{3}{5}} \Bigg|_0^{1-\varepsilon} + \lim_{\varepsilon \to 0+} \frac{5}{3}(x-1)^{\frac{3}{5}} \Bigg|_{1+\varepsilon}^{2} = 0 - \frac{5}{3}(-1)^{\frac{3}{5}} + \frac{5}{3} - 0 = \boxed{-\frac{5}{3}((-1)^{\frac{3}{5}} - 1)}$$

Primer 7.

Nadji površinu ograničenu krivom $y=\dfrac{x}{(1+x^2)^3}$ **i njenom asimptotom.**

Funkcija je svuda definisana.

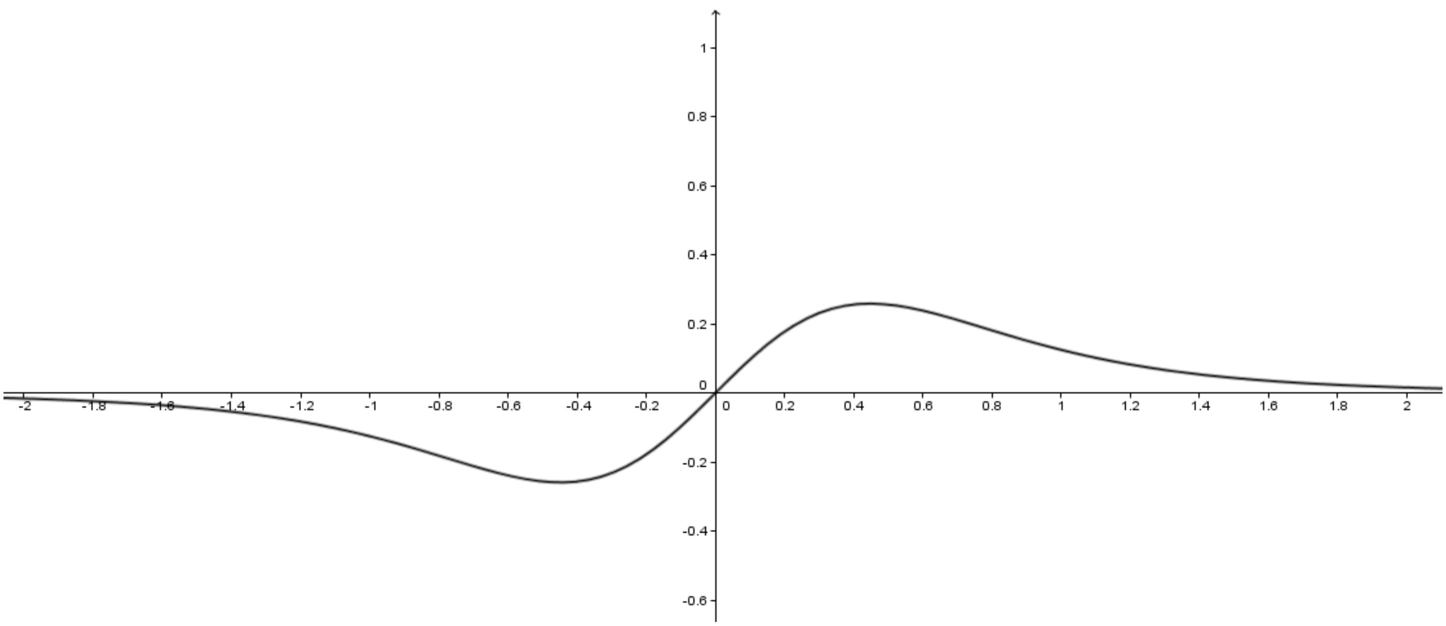
Nula funkcije je u nuli . x=0

Funkcija je neparna f(-x)=f(x)

Nema dakle vertikalnu asimptotu a kako je $\lim\limits_{x\to\pm\infty}\dfrac{x}{(1+x^2)^3}=0$ zaključujemo da je x osa horizontalna asimptota.

Ako profesor baš insistira, ispitajte i ekstreme....

Skica je:



Naći ćemo integral od 0 do beskonačnosti, pa to pomnožiti sa dva .

$$P=2\int_0^{\infty}\frac{x}{(1+x^2)^3}dx=2\lim_{b\to\infty}\int_0^{b}\frac{x}{(1+x^2)^3}dx$$

$$\int\frac{x}{(1+x^2)^3}dx=\left|\begin{array}{l}1+x^2=t\\ 2xdx=dt\\ xdx=\dfrac{dt}{2}\end{array}\right|=\int\frac{\frac{dt}{2}}{t^3}=\frac{1}{2}\int t^{-3}dt=\frac{1}{2}\frac{t^{-2}}{-2}=-\frac{1}{4t^2}=-\frac{1}{4(1+x^2)}$$

$$P=2\lim_{b\to\infty}\int_0^{b}\frac{x}{(1+x^2)^3}dx=2\cdot\lim_{b\to\infty}\left(-\frac{1}{4(1+x^2)}\right)\Bigg|_0^b=-\frac{1}{2}\left(\lim_{b\to\infty}\frac{1}{(1+b^2)}-1\right)=\frac{1}{2}$$