INTEGRALI ZADACI (V-DEO)

Integrali nekih funkcija koje sadrže kvadratni trinom ax^2+bx+c

Najpre ćemo proučiti integrale oblika: $\boxed{I_1=\int\frac{dx}{ax^2+bx+c}}$ i $\boxed{I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}}$

Kod njih se kvadratni trinom ax^2+bx+c svede na kanonični oblik pomoću formule:

$$\boxed{ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}}$$

naravno, možemo koristiti i dopunu do punog kvadrata, ko ne voli da pamti formulu.

Zatim uzimamo smenu: $\left|\begin{matrix} x+\frac{b}{2a}=t \\ dx=dt\end{matrix}\right|$, i dobijemo neki od tabličnih integrala.

$\boxed{I_1=\int\frac{dx}{ax^2+bx+c}}$ se može svesti na $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}arctg\frac{x}{a}+C$, $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

ili $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

$\boxed{I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}}$ se svodi najčešće na $\int\frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+C$ ili $\int\frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

$\boxed{primer\ 1.}$ $\int\frac{dx}{x^2-6x+13}=?$

$x^2-6x+13=$ Ovde je $a=1,\ b=-6,\ c=13$ pa to zamenimo u formulicu $a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$, dakle:

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}=1\left(x+\frac{-6}{2\cdot 1}\right)^2+\frac{4\cdot 1\cdot 13-(-6)^2}{4\cdot 1}=(x-3)^2+\frac{52-36}{4}=\boxed{(x-3)^2+4}$$

Lakše je naravno izvršiti dopunu do punog kvadrata, znate ono dodamo i oduzmemo onaj uz x podeljen sa 2 pa to na kvadrat. $\left(\frac{broj\ uz\ x}{2}\right)^2$

$x^2-6x+13=\underline{x^2-6x+9}-9+13=\boxed{(x-3)^2+4}$ Uz x je 6, pa dodajemo i oduzimamo $\left(\frac{6}{2}\right)^2=9$

Vraćamo se u integral:

$$\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+4} = \left| \begin{matrix} x-3=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{t^2+2^2}$$ (ovo je $\int \frac{1}{a^2+x^2}dx = \frac{1}{a} arctg \frac{x}{a} + C$ iz tabele)=

$$\frac{1}{2} arctg \frac{t}{2} + C = (vratimo \;\; smenu) = \boxed{\frac{1}{2} arctg \frac{x-3}{2} + C}$$

$\boxed{primer\ 2.}$ $\int \frac{dx}{\sqrt{2x^2-6x+5}} = ?$

$$2x^2-6x+5 = 2(x^2-3x+\frac{5}{2}) = 2(x^2-3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}) = 2[(x-\frac{3}{2})^2+\frac{1}{4}]$$

$$\int \frac{dx}{\sqrt{2x^2-6x+5}} = \int \frac{dx}{\sqrt{2[(x-\frac{3}{2})^2+\frac{1}{4}]}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x-\frac{3}{2})^2+\frac{1}{4}}} = \left| \begin{matrix} x-\frac{3}{2}=t \\ dx=dt \end{matrix} \right| = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2+(\frac{1}{2})^2}}$$

Upotrebimo iz tablice $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$

$$\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2+(\frac{1}{2})^2}} = \frac{1}{\sqrt{2}} \ln\left|t+\sqrt{t^2+(\frac{1}{2})^2}\right| + C = vratimo \;\; smenu = \boxed{\frac{1}{\sqrt{2}} \ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right| + C}$$

Kad znamo ova dva tipa integrala ,možemo naučiti i :

$$\boxed{I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx} \quad \text{i} \quad \boxed{I_4 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx}$$

Oni se radom svedu na prethodna dva integrala:

$I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$ se svede na integral $I_1 = \int \frac{dx}{ax^2+bx+c}$, dok se

$I_4 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ svede na integral $I_3 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$.

Postoje gotove formulice u kojima treba samo da uporedite i nadjete vrednosti za A,B,a,b i c.

Pazite: njih smete koristiti samo ako to odobrava vaš profesor! Mi ćemo vam pokazati i ceo postupak u slučaju da ne smete da koristite formule…

Formulice su:

$$I_2 = \frac{A}{2a}\ ln\ \left|ax^2+bx+c\right| + (B- \frac{Ab}{2a})\ I_1 + C \quad \text{i} \quad I_4 = \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{A\cdot b}{2a}\right) I_3 + C$$

$\boxed{primer\ 3.}$ $\int \frac{x+1}{x^2+x+1}dx = ?$

Ovo je očigledno integral tipa $I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$

Uporedjivanjem dobijamo da je : *A=1, B=1, a=1, b=1,c=1*

$$I_2 = \frac{A}{2a}\ln\left|ax^2+bx+c\right| + (B - \frac{Ab}{2a})I_1 + C$$

$$A=1, B=1, a=1, b=1, c=1$$

$$I_2 = \frac{1}{2\cdot 1}\ln\left|1x^2+1x+1\right| + (1 - \frac{1\cdot 1}{2\cdot 1})I_1 + C = \boxed{\frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{2}I_1 + C}$$

Sad imamo poso da rešimo integral tipa $I_1 = \int \frac{1}{x^2+x+1}dx$ i da njegovo rešenje posle vratimo u formulu.

$$I_1 = \int \frac{1}{x^2+x+1}dx = ?$$

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$I_1 = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \left| \begin{matrix} x+\frac{1}{2} = t \\ dx = dt \end{matrix} \right| = \int \frac{1}{t^2+(\frac{\sqrt{3}}{2})^2}dx = \frac{1}{\frac{\sqrt{3}}{2}}arctg\frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}arctg\frac{2(x+\frac{1}{2})}{\sqrt{3}} =$$

$$= \boxed{\frac{2}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}}$$

Vratimo se u formulu:

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{2}I_1 + C = \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C$$

$$\boxed{\frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C}$$

Kako bi ovaj integral rešavali da nismo smeli koristiti formulu?

$$\int \frac{x+1}{x^2+x+1}dx = ?$$

Ideja je da se izraz u brojiocu ***Ax+B*** **napravi da bude izvod izraza u imeniocu** ax^2+bx+c **.**

To možete uraditi tako što izvučete ispred integrala $\frac{A}{2a}$**.**

U našem primeru imamo x^2+x+1 u imeniocu, njegov izvod je $(x^2+x+1)`=2x+1$, što znači da u brojiocu treba da napravimo $2x+1$, odnosno da izvučemo $\frac{A}{2a}=\frac{1}{2}$ ispred integrala!

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx = \frac{1}{2}(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{1}{x^2+x+1}dx)$$

Sad se problem sveo na rešavanje dva integrala, gde prvi uvek radimo smenom, a drugi je tipa I_1.

$$\int \frac{2x+1}{x^2+x+1}dx = \left| \begin{matrix} x^2+x+1=t \\ (2x+1)dx=dt \end{matrix} \right| = \int \frac{1}{t}dt = \ln|t| = \ln|x^2+x+1|$$

Ovaj drugi smo već rešavali:

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \left| \begin{matrix} x+\frac{1}{2}=t \\ dx=dt \end{matrix} \right| = \int \frac{1}{t^2+(\frac{\sqrt{3}}{2})^2}dx = \frac{1}{\frac{\sqrt{3}}{2}} arctg \frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} arctg \frac{2(x+\frac{1}{2})}{\sqrt{3}} =$$

$$= \frac{2}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}}$$

Vratimo se na zadatak :

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{1}{x^2+x+1}dx) =$$

$$= \frac{1}{2}(\ln|x^2+x+1| + \frac{2}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}}) + C$$

$$= \boxed{\frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}} + C}$$

$\boxed{primer\ 4.}$ $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$

I način (uz pomoć formule)

$$I_4 = \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{A\cdot b}{2a}\right)I_3$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$$

$$A=5, B=3, a=1, b=4, c=10$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{1}\sqrt{x^2+4x+10} + \left(3 - \frac{5\cdot 4}{2\cdot 1}\right)\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= 5\cdot\sqrt{x^2+4x+10} + (-7)\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= 5\cdot\sqrt{x^2+4x+10} - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Da rešimo ovaj integral posebno, pa ćemo vratiti njegovo rešenje...

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx =$$

$$x^2+4x+10 = x^2+4x+4-4+10 = (x+2)^2+6$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{\sqrt{(x+2)^2+6}}dx = \begin{vmatrix} x+2=t \\ dx=dt \end{vmatrix} = \int \frac{1}{\sqrt{t^2+6}}dx = koristimo: \boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C}$$

$$= \ln\left|t+\sqrt{t^2+6}\right| = \ln\left|x+2+\sqrt{x^2+4x+10}\right|$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \boxed{5\cdot\sqrt{x^2+4x+10} - 7\cdot\ln\left|x+2+\sqrt{x^2+4x+10}\right| + C}$$

II način (direktno, bez upotrebe formulice)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$$

Kako je izvod $(x^2+4x+10)` = 2x+4 = 2(x+2)$ u brojiocu mora biti napravljeno to.

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{5(x+\frac{3}{5})}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2-2+\frac{3}{5}}{\sqrt{x^2+4x+10}}dx =$$

$$= 5\cdot(\int \frac{x+2}{\sqrt{x^2+4x+10}}dx + \int \frac{-2+\frac{3}{5}}{\sqrt{x^2+4x+10}}dx)$$

$$= 5\cdot(\int \frac{x+2}{\sqrt{x^2+4x+10}}dx + \int \frac{-\frac{7}{5}}{\sqrt{x^2+4x+10}}dx)$$

$$= 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Sad radimo ova dva integrala (drugi smo već rešavali kod prvog načina).

$$\int \frac{x+2}{\sqrt{x^2+4x+10}}dx = \left| \begin{array}{l} x^2+4x+10=t^2 \\ (2x+4)dx = 2tdt \\ \cancel{2}(x+2)dx = \cancel{2}tdt \\ (x+2)dx = tdt \end{array} \right| = \int \frac{\cancel{t}dt}{\cancel{t}} = \int dt = t = \sqrt{x^2+4x+10}$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx =$$

$$x^2+4x+10 = x^2+4x+4-4+10 = (x+2)^2+6$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{\sqrt{(x+2)^2+6}}dx = \left| \begin{array}{l} x+2=t \\ dx=dt \end{array} \right| = \int \frac{1}{\sqrt{t^2+6}}dx = koristimo: \boxed{\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C}$$

$$= \ln\left|t+\sqrt{t^2+6}\right| = \ln\left|x+2+\sqrt{x^2+4x+10}\right|$$

Vratimo se u zadatak:

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \boxed{5\cdot\sqrt{x^2+4x+10} - 7\cdot\ln\left|x+2+\sqrt{x^2+4x+10}\right| + C}$$

$\boxed{primer\ 5.}$ $\int \frac{2x+7}{x^2+x-2}dx = ?$

Ovaj primer vam navodimo jer trebate voditi računa o polinomu u imeniocu!

Rekli bi da je ovo integral tipa $I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$ i radili bi:

$$\int \frac{2x+7}{x^2+x-2}dx = \int \frac{2x+1+6}{x^2+x-2}dx = \int \frac{2x+1}{x^2+x-2}dx + \int \frac{6}{x^2+x-2}dx =$$

Rešimo ova dva integrala posebno, pa ćemo zameniti njihova rešenja...

$$\int \frac{2x+1}{x^2+x-2}dx = \begin{vmatrix} x^2+x-2=t \\ (2x+1)dx=dt \end{vmatrix} = \int \frac{dt}{t} = \ln|t| = \ln\left|x^2+x-2\right|$$

$$\int \frac{6}{x^2+x-2}dx = 6\int \frac{1}{x^2+x-2}dx =$$

$$x^2+x-2 = x^2+x+\frac{1}{4}-\frac{1}{4}-2 = (x+\frac{1}{2})^2 - \frac{9}{4}$$

$$6\int \frac{1}{x^2+x-2}dx = 6\int \frac{1}{(x+\frac{1}{2})^2-\frac{9}{4}}dx = \begin{vmatrix} x+\frac{1}{2}=t \\ dx=dt \end{vmatrix} = 6\int \frac{1}{t^2-\left(\frac{3}{2}\right)^2}dx = koristimo: \boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}$$

$$= 6\cdot\frac{1}{2\cdot\frac{3}{2}}\ln\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right| = 2\ln\left|\frac{x+\frac{1}{2}-\frac{3}{2}}{x+\frac{1}{2}+\frac{3}{2}}\right| = 2\ln\left|\frac{x-1}{x+2}\right|$$

vratimo rešenja:

$$\int \frac{2x+7}{x^2+x-2}dx = \int \frac{2x+1}{x^2+x-2}dx + \int \frac{6}{x^2+x-2}dx =$$

$$= \ln\left|x^2+x-2\right| + 2\ln\left|\frac{x-1}{x+2}\right| + C$$

$$= \ln|(x-1)(x+2)| + \ln\left|\frac{x-1}{x+2}\right|^2 + C$$

$$= \ln\left|(x-1)\cancel{(x+2)}\cdot\frac{(x-1)^2}{(x+2)^{\cancel{2}}}\right| + C$$

$$\boxed{= \ln\left|\frac{(x-1)^3}{(x+2)}\right| + C}$$

Nije bilo lako rešiti ga, priznaćete...

Nismo razmišljali jednu drugu stvar: **Da li je ovaj zadatak mogo da se uradi kao integracija racionalne funkcije?**

Proverimo da li polinom u imeniocu može da se rastavi na činioce...

$$x^2+x-2=0 \to x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \to x_1=1, x_2=-2 \qquad \textbf{MOŽE!}$$

Lakše je raditi (bar nama):

$$\int \frac{2x+7}{x^2+x-2}dx=?$$

$$\frac{2x+7}{x^2+x-2}=\frac{2x+7}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots/\bullet(x-1)(x+2)$$

$$2x+7=A(x+2)+B(x-1)$$

$$2x+7=Ax+2A+Bx-B$$

$$2x+7=x(A+B)+2A-B$$

$$\textit{uporedjujemo}$$

$$A+B=2$$

$$\underline{2A-B=7}$$

$$3A=9\to \boxed{A=3}\to \boxed{B=-1}$$

$$\frac{2x+7}{(x-1)(x+2)}=\frac{3}{x-1}+\frac{-1}{x+2}=\frac{3}{x-1}-\frac{1}{x+2}$$

$$\int \frac{2x+7}{(x-1)(x+2)}dx=\int \frac{3}{x-1}dx-\int \frac{1}{x+2}dx=3\ln|x-1|-\ln|x+2|+C$$

$$=\ln|x-1|^3-\ln|x+2|+C$$

$$\boxed{=\ln\left|\frac{(x-1)^3}{x+2}\right|+C}$$

Naš savet je dakle da proverite da li je kvadratna jednačina u imeniocu rešiva i da ako jeste radite integral kao integraciju racionalne funkcije.

Ako kvadratna nije rešiva, radite ga kao integral tipa $I_2=\int \frac{Ax+B}{ax^2+bx+c}dx$.

Videli ste da su ispala ista rešenja.

Uostalom, odlučite sami, šta vama više odgovara ili kako pak komanduje profesor...

Sledeći tip integrala je $\boxed{\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}}$

Ovi integrali se smenom: $\left| \begin{array}{l} mx+n=\frac{1}{t} \\ mdx=-\frac{1}{t^2}dt \\ dx=-\frac{1}{m\cdot t^2}dt \end{array} \right|$, svedu na integral tipa $I_3=\int \frac{dx}{\sqrt{ax^2+bx+c}}$

$\boxed{primer\ 6.}$ $\int \frac{dx}{x\cdot\sqrt{x^2-4x+1}}=?$

Najpre uzimamo smenu $x=\frac{1}{t}$ kojom svodimo dati integral na tip I_3.

$$\int \frac{dx}{x\cdot\sqrt{x^2-4x+1}}=\left| \begin{array}{l} x=\frac{1}{t} \\ dx=-\frac{1}{t^2}dt \end{array} \right|=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\cdot\sqrt{(\frac{1}{t})^2-4\cdot\frac{1}{t}+1}}=\int \frac{-\frac{1}{t^{\cancel{2}}}dt}{\frac{1}{\cancel{t}}\cdot\sqrt{\frac{1}{t^2}-\frac{4}{t}+1}}=\int \frac{-\frac{1}{t}dt}{\sqrt{\frac{1-4t+t^2}{t^2}}}=$$

$$=\int \frac{-\frac{1}{\cancel{t}}dt}{\frac{1}{\cancel{t}}\sqrt{t^2-4t+1}}=\int \frac{-dt}{\sqrt{t^2-4t+1}}=-\int \frac{dt}{\sqrt{t^2-4t+1}}$$

$$t^2-4t+1=t^2-4t+4-4+1=(t-2)^2-3$$

$$-\int \frac{dt}{\sqrt{t^2-4t+1}}=-\int \frac{dt}{\sqrt{(t-2)^2-3}}=\left| \begin{array}{l} t-2=z \\ dt=dz \end{array} \right|=-\int \frac{dt}{\sqrt{z^2-3}}=koristimo:\int \frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C, pa\ je$$

$$=-\ln\left|z+\sqrt{z^2-3}\right|=vratimo\ smenu=-\ln\left|t-2+\sqrt{t^2-4t+1}\right|+C$$

Moramo da vratimo i prvu smenu:

$$=-\ln\left|t-2+\sqrt{t^2-4t+1}\right|+C=\boxed{-\ln\left|\frac{1}{x}-2+\sqrt{(\frac{1}{x})^2-4\cdot\frac{1}{x}+1}\right|+C}$$

<u>Metoda neodredjenih koeficijenata</u> (metoda Ostrogradskog)

Ovom metodom se rešavaju integrali tipa $\boxed{\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx}$ gde je u brojiocu podintegralne funkcije imamo polinom n-tog stepena.

Postupak rada je sledeći:

- **postavimo jednačinu**

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

Ovde je $Q_{n-1}(x)$ polinom (n-1) vog stepena sa neodredjenim koeficijentima.

- **ovu jednačinu diferenciramo**
- **zatim sve pomnožimo sa** $\sqrt{ax^2+bx+c}$
- **sa obe strane dobijamo polinome reda *n*, pa neodredjene koeficijente odredjujemo izjednačavanjem koeficijenata uz iste stepene x-a.**

Kako je polinom $P_n(x)$ u zadacima najčešće drugog stepena početna jednačina će biti:

$$\int \frac{mx^2+px+r}{\sqrt{ax^2+bx+c}}dx = (Ax+B)\sqrt{ax^2+bx+c} + \lambda\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

Ali, najbolje da to vidimo na primeru:

$\boxed{primer\ 7.}$ $\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = ?$

Postavimo jednačinu:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = (Ax+B)\cdot\sqrt{x^2+2x+2} + \lambda\cdot\int\frac{dx}{\sqrt{x^2+2x+2}} \to diferenciramo$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = (Ax+B)'\cdot\sqrt{x^2+2x+2} + (\sqrt{x^2+2x+2})'\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{2\sqrt{x^2+2x+2}}(x^2+2x+2)'\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{2\sqrt{x^2+2x+2}}(2x+2)\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{\cancel{2}\sqrt{x^2+2x+2}}\cancel{2}(x+1)\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{(x+1)\cdot(Ax+B)}{\sqrt{x^2+2x+2}} + \frac{\lambda}{\sqrt{x^2+2x+2}} \to \ldots\ldots\ldots\ldots\ldots\ldots / \bullet\sqrt{x^2+2x+2}$$

$$2x^2+3x = A(x^2+2x+2)+(x+1)\cdot(Ax+B)+\lambda$$

Sada uporedjujemo koeficijente:

$$2x^2+3x = A(x^2+2x+2)+(x+1)\cdot(Ax+B)+\lambda$$

$$2x^2+3x = \underline{Ax^2} + \overline{2Ax} + 2A + \underline{Ax^2} + \overline{Bx} + \overline{Ax} + B + \lambda$$

$$2x^2+3x = 2Ax^2 + x(3A+B) + 2A + B + \lambda \to uporedjujemo$$

$$2A = 2 \to \boxed{A=1}$$

$$3A+B=3$$

$$\underline{2A+B+\lambda=0}$$

$$3A+B=3 \to 3\cdot 1+B=3 \to \boxed{B=0}$$

$$2A+B+\lambda=0 \to 2+0+\lambda=0 \to \boxed{\lambda=-2}$$

Vratimo se u početnu jednačinu:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = (Ax+B)\cdot\sqrt{x^2+2x+2} + \lambda\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$= (1x+0)\cdot\sqrt{x^2+2x+2} - 2\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$= x\cdot\sqrt{x^2+2x+2} - 2\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

Da rešimo posebno ovaj integral...

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{x^2+2x+1+1}} = \int \frac{dx}{\sqrt{(x+1)^2+1}} = \begin{vmatrix} x+1=t \\ dx=dt \end{vmatrix} = \int \frac{dx}{\sqrt{t^2+1}} =$$

$$upotrebimo: \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$$

$$= \ln\left|t+\sqrt{t^2+1}\right| + C = \boxed{\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C}$$

Konačno, rešenje će biti:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}} dx = x\cdot\sqrt{x^2+2x+2} - 2\cdot\int \frac{dx}{\sqrt{x^2+2x+2}}$$

$$= \boxed{x\cdot\sqrt{x^2+2x+2} - 2\cdot\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C}$$

$\boxed{primer\ 8.}$

$$\int \sqrt{a^2-x^2}\,dx = ?$$

Ako se sećate, ovaj integral smo rešavali u fajlu parcijalna integracija. Tada smo rekli da on može da se rešava na više načina. Evo kako bi išlo rešavanje metodom Ostrogradskog.

Naravno, opet racionalizacijom malo prepravimo podintegralnu funkciju...

$$\frac{\sqrt{a^2-x^2}}{1}\cdot\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} = \frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$\int \sqrt{a^2-x^2}\,dx = \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx$$

Sada je ovo oblik koji nam treba…

$$\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx = (Ax+B)\cdot\sqrt{a^2-x^2} + \lambda\cdot\int\frac{dx}{\sqrt{a^2-x^2}} \rightarrow \textit{diferenciramo}$$

$$\frac{a^2-x^2}{\sqrt{a^2-x^2}} = A\sqrt{a^2-x^2} + \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}(Ax+B) + \frac{\lambda}{\sqrt{a^2-x^2}}$$

$$\frac{a^2-x^2}{\sqrt{a^2-x^2}} = A\sqrt{a^2-x^2} + \frac{-x}{\sqrt{a^2-x^2}}(Ax+B) + \frac{\lambda}{\sqrt{a^2-x^2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots /\bullet\sqrt{a^2-x^2}$$

$$a^2-x^2 = A(a^2-x^2) - x(Ax+B) + \lambda$$

$$a^2-x^2 = Aa^2 - Ax^2 - Ax^2 - Bx + \lambda$$

$$a^2-x^2 = -2Ax^2 - Bx + Aa^2 + \lambda$$

uporedjujemo

$$-2A = -1$$

$$-B = 0 \rightarrow \boxed{B=0}$$

$$\underline{Aa^2 + \lambda = a^2}$$

Rešavamo ovaj sistemčić

$$\boxed{A=\frac{1}{2}} \rightarrow \frac{1}{2}a^2 + \lambda = a^2 \rightarrow \boxed{\lambda = \frac{1}{2}a^2}$$

Vratimo se u postavku…

$$\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx = (Ax+B)\cdot\sqrt{a^2-x^2} + \lambda\cdot\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx = \frac{1}{2}x\cdot\sqrt{a^2-x^2} + \frac{1}{2}a^2\cdot\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$\boxed{\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx = \frac{1}{2}x\cdot\sqrt{a^2-x^2} + \frac{1}{2}a^2\cdot\arcsin\frac{x}{a} + C}$$