GRANIČNE VREDNOSTI FUNKCIJA – ZADACI II deo

U sledećim zadacima ćemo koristiti poznatu graničnu vrednost:

$\boxed{\lim\limits_{x\to 0}\frac{\sin x}{x}=1}$ ali i manje “varijacije’’

$\lim\limits_{x\to 0}\frac{\sin ax}{ax}=1$ i $\lim\limits_{x\to 0}\frac{\sin^n ax}{(ax)^n}=1$

Zadaci:

1) Odrediti sledeće granične vrednosti:

a) $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$;

b) $\lim\limits_{x\to 0}\frac{tgx}{x}$;

v) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$;

g) $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}$;

Rešenja:

a) $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$; (i gore i dole dodamo 4) $=\lim\limits_{x\to 0}\frac{4\sin 4x}{4x}=4\cdot\boxed{\lim\limits_{x\to 0}\frac{\sin 4x}{4x}}=4\cdot 1=4$

Ovde smo “ napravili” i upotrebili da je $\lim\limits_{x\to 0}\frac{\sin ax}{ax}=1$

b) $\lim\limits_{x\to 0}\frac{tgx}{x}=\lim\limits_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim\limits_{x\to 0}\boxed{\frac{\sin x}{x}}\cdot\frac{1}{\cos x}=\lim\limits_{x\to 0}1\cdot\frac{1}{\cos x}$

$=1\cdot\lim\limits_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{\cos 0}=1\cdot\frac{1}{1}=1$

v) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=$ iskoristićemo formulu iz trigonometrije: $1-\cos x=2\sin^2\frac{x}{2}$

$=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=(\text{ dodamo } 4)=\lim\limits_{x\to 0}2\cdot\frac{\sin^2\frac{x}{2}}{4\frac{x^2}{4}}=\frac{2}{4}\cdot\boxed{\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}=\frac{1}{2}\cdot 1=\frac{1}{2}$

g) $\lim\limits_{x\to a} = \dfrac{\sin x - \sin a}{x-a} =$ iskoristićemo formulu (pogledaj PDF fajl iz II godine)

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= \lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \text{malo prisredimo...}$$

$$= \lim_{x\to a}\cos\frac{x+a}{2}\cdot\boxed{\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}} = \lim_{x\to a}\cos\frac{x+a}{2}\cdot 1 =$$

$$= \cos\frac{a+a}{2} = \cos\frac{2a}{2} = \cos a$$

2) Izračunati sledeće granične vrednosti:

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1}$;

b) $\lim\limits_{x\to \pi}\dfrac{\cos\frac{x}{2}}{x-\pi}$;

v) $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$;

a)

$$\lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1} = \text{najpre racionalizacija}$$

$$= \lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x+1-1} = \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x}$$

sad i gore i dole dodamo 4

$$= \lim_{x\to 0}\frac{4\sin 4x\left(\sqrt{x+1}+1\right)}{4x} = \lim_{x\to 0}4\boxed{\frac{\sin 4x}{4x}}\left(\sqrt{x+1}+1\right) = \lim_{x\to 0}4\cdot 1\cdot\left(\sqrt{x+1}+1\right) =$$

$$= 4\left(\sqrt{0+1}+1\right) = 4\cdot 2 = 8$$

b)

$$\lim_{x\to\pi}\frac{\cos\frac{x}{2}}{x-\pi}=\text{ovde ćemo najpre uzeti smenu: } x-\pi=t,\text{, pa kad } x\to\pi\text{, onda}$$

$t\to\pi-\pi=0$, dakle $\boxed{t\to 0}$

$$\lim_{t\to0}\frac{\cos\frac{t+\pi}{2}}{t}=\lim_{t\to0}\frac{\cos\left(\frac{\pi}{2}+\frac{t}{2}\right)}{t}=\lim_{t\to0}\frac{-\sin\frac{t}{2}}{t}\quad(\text{ jer je } \cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha\ )$$

$$=-\lim_{t\to0}\frac{\sin\frac{t}{2}}{2\cdot\frac{t}{2}}=-\lim_{t\to0}\frac{1}{2}\boxed{\frac{\sin\frac{t}{2}}{\frac{t}{2}}}=-\frac{1}{2}\cdot1=-\frac{1}{2}$$

v)

$$\lim_{x\to1}\frac{\sin(1-x)}{\sqrt{x}-1}=\text{ najpre racionalizacija}$$

$$\lim_{x\to1}\frac{\sin(1-x)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to1}\frac{\sin(1-x)(\sqrt{x}+1)}{x-1}=\text{ sada smena } x-1=t,\text{ kad } x\to1$$

tad $t\to0$

$$=\lim_{t\to0}\frac{\sin(-t)\left(\sqrt{t+1}+1\right)}{t}=\lim_{t\to0}\frac{-\sin(t)\left(\sqrt{t+1}+1\right)}{t}=-\lim_{t\to0}\boxed{\frac{\sin t}{t}}\left(\sqrt{t+1}+1\right)$$

$$=-\lim_{t\to0}1\cdot\left(\sqrt{t+1}+1\right)=-(1+1)=-2$$

U sledećim zadacima ćemo koristiti:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e}\qquad\text{i}\qquad\lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax}=e$$

Još nam treba i činjenica da je e^x neprekidna funkcija i važi:

$$\boxed{\lim_{x\to a}e^{f(x)}=e^{\lim\limits_{x\to a}f(x)}}$$

3) Odrediti sledeće granične vrednosti:

a) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}$;

b) $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$;

c) $\lim\limits_{x\to\infty}x(\ln(x+1)-\ln x)$;

Rešenja:

a)

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \text{ovde gde je 3 mora biti 1, pa ćemo 3 ‘spustiti’ ispod x}$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x = \text{ sad kod } x \text{ u eksponentu pomnožimo i podelimo sa } 3$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x = = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot 3} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}}^{3} = e^3$$

b)

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x = \textbf{trik}\text{: u zagradi ćemo dodati 1 i oduzeti } 1 =$$

$$\lim_{x\to\infty}\left(1+\frac{x+1}{x-1}-1\right)^x = \lim_{x\to\infty}\left(1+\frac{x+1-1(x-1)}{x-1}\right)^x = \lim_{x\to\infty}\left(1+\frac{x+1-x+1}{x-1}\right)^x$$

$$= \lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} =$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}}^{\frac{2x}{x-1}} = \lim_{x\to\infty} e^{\frac{2x}{x-1}} = e^{\lim_{x\to\infty}\frac{2x}{x-1}} = e^2$$

v)

$$\lim_{x\to\infty} x\cdot(\ln(x+1)-\ln x) = \lim_{x\to\infty}[x\cdot\ln\frac{x+1}{x}] = \lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right)^x =$$

(pošto je ln neprekidna funkcija i ona može da zameni mesto sa lim)

$$\ln\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = \ln\lim_{x\to\infty}\left(\frac{x}{x}+\frac{1}{x}\right)^x = \ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \ln e = 1$$

Ovde smo koristili pravila(pogledaj II godina logaritmi): lnA - $lnB = \ln\frac{A}{B}$ i

$n\cdot\ln A = \ln A^n$

4) Odrediti sledeće granične vrednosti:

a) $\lim\limits_{x\to 0}(1+3tg^2x)^{ctg^2x} = ?$

b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = ?$

__Rešenja:__

a) $\lim\limits_{x\to 0}(1+3tg^2x)^{ctg^2x} = ?$

$$\lim_{x\to 0}(1+3tg^2x)^{ctg^2x} = \lim_{x\to 0}\left(1+3\cdot\frac{1}{ctg^2x}\right)^{ctg^2x} = \lim_{x\to 0}\left(1+\frac{1}{\frac{ctg^2x}{3}}\right)^{ctg^2x} =$$

$$\lim_{x\to 0}\left(1+\frac{1}{\frac{ctg^2x}{3}}\right)^{ctg^2x\cdot\frac{3}{3}} = \lim_{x\to 0}\left(1+\frac{1}{\frac{ctg^2x}{3}}\right)^{\frac{ctg^2x}{3}\cdot 3} = \boxed{\lim_{x\to 0}\left(1+\frac{1}{\frac{ctg^2x}{3}}\right)^{\frac{ctg^2x}{3}}}^{3} = e^3$$

b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = ?$

Najpre ćemo dodati i oduzeti jedinicu…

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0}(1+\cos x-1)^{\frac{1}{\sin^2 x}}$$

Dalje moramo upotrebiti formulicu: $1-\cos x = 2\sin^2\frac{x}{2}$

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0}(1+\cos x-1)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0}(1-(1-\cos x))^{\frac{1}{\sin^2 x}} = \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0}\left(1\boxed{-}\frac{1}{\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}} =$$

$$= \lim_{x\to 0}\left(1+\frac{1}{\boxed{-}\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}} = \{formula\ \ \sin^2 x = 4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\} = \lim_{x\to 0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}} = \lim_{x\to 0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{-\frac{1}{\sin^2\frac{x}{2}}\cdot\frac{-1}{4\cos^2\frac{x}{2}}} =$$

$$= \boxed{\lim_{x\to 0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{-\frac{1}{2\sin^2\frac{x}{2}}}}^{\frac{-1}{2\cos^2\frac{x}{2}}} = e^{\lim\limits_{x\to 0}\frac{-1}{2\cos^2\frac{x}{2}}} = e^{-\frac{1}{2}}$$

Ko je upoznat sa Lopitalovom teoremom može ove zadačiće rešavati i na drugi način:

a) $\lim\limits_{x\to 0}(1+3tg^2x)^{ctg^2x}=?$

Ceo limes obeležimo sa nekim slovom, recimo A i **elenujemo** ga:

$\lim\limits_{x\to 0}(1+3tg^2x)^{ctg^2x}=A............/\ln$

$\ln\lim\limits_{x\to 0}(1+3tg^2x)^{ctg^2x}=\ln A$

$\lim\limits_{x\to 0}\ln(1+3tg^2x)^{ctg^2x}=\ln A$

$\lim\limits_{x\to 0}ctg^2x\cdot\ln(1+3tg^2x)=\ln A$

$\lim\limits_{x\to 0}\dfrac{1}{tg^2x}\ln(1+3tg^2x)=\ln A$

$\lim\limits_{x\to 0}\dfrac{\ln(1+3tg^2x)}{tg^2x}=\ln A$ sad na levoj strani upotrebljavamo Lopitalovu teoremu

$$\lim_{x\to 0}\frac{\frac{1}{1+3tg^2x}\cdot 3\cdot\cancel{2tgx\cdot\frac{1}{\cos^2x}}}{\cancel{2tgx\cdot\frac{1}{\cos^2x}}}=\ln A$$

$$\lim_{x\to 0}\frac{3}{1+3tg^2x}=\ln A\to\frac{3}{1+3tg^2 0}=\ln A\to\frac{3}{1}=\ln A\to\ln A=3\to\boxed{A=e^3}$$

b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2x}}=?$

$\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2x}}=A............/\ln$

$\ln\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2x}}=\ln A$

$\lim\limits_{x\to 0}\ln(\cos x)^{\frac{1}{\sin^2x}}=\ln A$

$\lim\limits_{x\to 0}\dfrac{1}{\sin^2x}\ln(\cos x)\ =\ln A$

$\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{\sin^2x}=\ln A$ na levoj strani Lopital...

$$\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\cancel{\sin x})}{2\cancel{\sin x}\cos x}=\ln A$$

$$\lim_{x\to 0}\frac{-1}{2\cos^2x}=\ln A\to\frac{-1}{2\cos^2 0}=\ln A\to\frac{-1}{2\cdot 1}=\ln A\to\frac{-1}{2}=\ln A\to\boxed{A=e^{-\frac{1}{2}}}$$

Vi naravno radite kako zahteva vaš profesor...

Kao što vidite, Lopitalova teorema je **elegantan** način da se dodje do rešenja kod odredjivanja graničnih vrednosti funkcija. **Ali pazite**, ona radi samo u situacijama $\frac{0}{0}$ i $\frac{\infty}{\infty}$.

5) Odrediti sledeće granične vrednosti:

a) $\lim\limits_{x\to 0} x^2 \ln x$

b) $\lim\limits_{x\to 0} x \cdot ctg 2x$

<u>Rešenja:</u>

a) $\lim\limits_{x\to 0} x^2 \ln x$

Ako zamenimo da x teži nuli , dobijamo : $\lim\limits_{x\to 0} x^2 \ln x = 0^2 \cdot \ln 0 = 0 \cdot (-\infty)$

Ovo je neodredjen izraz a **ne smemo** koristiti Lopitalovu teoremu **.**

Šta uraditi?

Moramo prepraviti funkciju od koje tražimo limes da bude oblika $\frac{0}{0}$ ili $\frac{\infty}{\infty}$.

$\lim\limits_{x\to 0} x^2 \ln x = \lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{x^2}} =$ ako ovde zamenimo da x teži nuli , dobijamo:

$\lim\limits_{x\to 0} x^2 \ln x = \lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{x^2}} = \frac{\ln 0}{\frac{1}{0^2}} = \frac{-\infty}{\infty}$ **,** pa možemo koristiti Lopitala…

$$\lim_{x\to 0} x^2 \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0} \frac{(\ln x)`}{(\frac{1}{x^2})`} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0} \frac{x^{\cancel{3}}}{-2\cancel{x}} = \lim_{x\to 0} \frac{x^2}{-2} = 0$$

b) $\lim\limits_{x\to 0} x \cdot ctg 2x$

Sličan trik kao u prethodnom primeru…

$$\lim_{x\to 0} x\cdot ctg2x = \lim_{x\to 0}\frac{ctg2x}{\frac{1}{x}} = (\frac{\infty}{\infty}) = \lim_{x\to 0}\frac{(ctg2x)`}{(\frac{1}{x})`} = \lim_{x\to 0}\frac{-\frac{1}{\sin^2 2x}\cdot 2}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{2x^2}{\sin^2 2x} = (\frac{0}{0})$$

Opet koristimo Lopitalovu teoremu…

$$\lim_{x\to 0}\frac{2x^2}{\sin^2 2x} = (\frac{0}{0}) = \lim_{x\to 0}\frac{(2x^2)`}{(\sin^2 2x)`} = \lim_{x\to 0}\frac{4x}{2\sin 2x\cdot\cos 2x\cdot 2} = \lim_{x\to 0}\frac{\cancel{4}x}{\cancel{4}\sin 2x\cdot\cos 2x} = \lim_{x\to 0}\frac{x}{\sin 2x\cdot\cos 2x} = (\frac{0}{0})$$

Auuu, opet Lopital…

$$\lim_{x\to 0}\frac{x}{\sin 2x\cdot\cos 2x} = (\frac{0}{0}) = \lim_{x\to 0}\frac{1}{[(\sin 2x)`\cdot\cos 2x + \sin 2x\cdot(\cos 2x)`]} =$$

$$\lim_{x\to 0}\frac{1}{[\cos 2x\cdot 2\cdot\cos 2x + \sin 2x\cdot(-\sin 2x)\cdot 2]} = \lim_{x\to 0}\frac{1}{2\cdot\cos^2 2x - 2\sin^2 2x} = \frac{1}{2\cdot\cos^2 2\cdot 0 - 2\sin^2 2\cdot 0} =$$

$$= \frac{1}{2\cdot\cos^2 0 - 2\sin^2 0} = \frac{1}{2\cdot 1 - 0} = \frac{1}{2}$$

Ovaj zadatak baš ispade težak, zar ne?

Al to je zato što ne razmišljamo, već odmah krenemo u rad...

Evo kako bi moglo prostije:

$$\lim_{x\to 0} x\cdot ctg2x = \lim_{x\to 0} x\cdot\frac{1}{tg2x} = \lim_{x\to 0}\frac{x}{tg2x} = (\frac{0}{0}) = \lim_{x\to 0}\frac{1}{\frac{1}{\cos^2 2x}\cdot 2} = \lim_{x\to 0}\frac{\cos^2 2x}{2} = \frac{\cos^2 0}{2} = \frac{1}{2}$$

Dakle, prvo pogledajte malo zadatak, analizirajte, pa onda krenite n