Nizovi-zadaci III deo

1. Odrediti granične vrednosti sledećih nizova:

a) $\lim\limits_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+3)!}$

b) $\lim\limits_{n\to\infty}\frac{(n+1)!-(n+3)!}{(n+3)!}$

c) $\lim\limits_{n\to\infty}\frac{(2n)!!+(2n+2)!!}{(n^2+1)\cdot(2n-2)!!}$

<u>Rešenja:</u>

Kod faktorijela nam je ideja da veći faktorijel razbijemo do manjeg i potom skratimo.

a)

$$\lim_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+3)!}=\text{Ovde }(\text{n+2})!\text{ i }(\text{n+3})!\text{ razbijamo do }(\text{n+1})!$$

$$=\lim_{n\to\infty}\frac{(n+2)(n+1)!+(n+1)!}{(n+3)(n+2)(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+1)![n+2+1]}{(n+3)(n+2)(n+1)!}$$

$$=\lim_{n\to\infty}\frac{\cancel{(n+1)!}[n+2+1]}{(n+3)(n+2)\,\cancel{(n+1)!}}$$

$$=\lim_{n\to\infty}\frac{n+3}{n^2+5n+6}=0$$

b)

$$\lim_{n\to\infty}\frac{(n+1)!-(n+3)!}{(n+3)!}=$$

$$\lim_{n\to\infty}\frac{(n+1)!-(n+3)(n+2)(n+1)!}{(n+3)(n+2)(n+1)!}=$$

$$\lim_{n\to\infty}\frac{(n+1)![1-(n+3)(n+2)]}{(n+3)(n+2)(n+1)!}=$$

$$\lim_{n\to\infty}\frac{\cancel{(n+1)!}[1-(n+3)(n+2)]}{(n+3)(n+2)\,\cancel{(n+1)!}}=$$

$$\lim_{n\to\infty}\frac{1-(n^2+5n+6)}{n^2+5n+6}=$$

$$\lim_{n\to\infty}\frac{-n^2-5n-5}{n^2+5n+6}=\frac{-1}{1}=-1$$

c)

Slična ideja je i kod duplog faktorijela….

Pazite, ovde uzimamo svaki drugi, recimo:

$$12!! = 12*10*8*6*4*2$$

Da krenemo sa radom:

$$\lim_{n\to\infty}\frac{(2n)!! + (2n+2)!!}{(n^2+1)\cdot(2n-2)!!} =$$

$$\lim_{n\to\infty}\frac{(2n)(2n-2)!! + (2n+2)(2n)(2n-2)!!}{(n^2+1)\cdot(2n-2)!!} =$$

$$\lim_{n\to\infty}\frac{(2n-2)!![2n+(2n+2)(2n)]}{(n^2+1)\cdot(2n-2)!!} =$$

$$\lim_{n\to\infty}\frac{\cancel{(2n-2)!!}[2n+(2n+2)(2n)]}{(n^2+1)\cdot\cancel{(2n-2)!!}} =$$

$$\lim_{n\to\infty}\frac{4n^2+6n}{n^2+1} = \frac{4}{1} = 4$$

2. Odrediti granične vrednosti sledećih nizova:

a) $\lim_{n\to\infty}\frac{2^n+3^n}{2^{n+1}+3^{n+1}}$

b) $\lim_{n\to\infty}\frac{5\cdot 2^n - 3\cdot 5^{n+1}}{100\cdot 2^n + 2\cdot 5^n}$

<u>Rešenja:</u>

Ovde nam je ideja da i gore i dole izvučemo kao zajednički onaj koji ima veći eksponent.

a)

$$\lim_{n\to\infty}\frac{2^n+3^n}{2^{n+1}+3^{n+1}}=$$

$$\lim_{n\to\infty}\frac{2^n+3^n}{2^n\cdot 2+3^n\cdot 3}=$$

$$\lim_{n\to\infty}\frac{3^n(\frac{2^n}{3^n}+1)}{3^n(\frac{2^n}{3^n}\cdot 2+3)}=$$

$$\lim_{n\to\infty}\frac{\cancel{3^n}((\frac{2}{3})^n+1)}{\cancel{3^n}((\frac{2}{3})^n\cdot 2+3)}=\lim_{n\to\infty}\frac{\overset{\text{teži } 0}{\boxed{(\frac{2}{3})^n}}+1}{\underset{\text{teži } 0}{\boxed{(\frac{2}{3})^n}}\cdot 2+3}=\frac{1}{3}$$

b)

$$\lim_{n\to\infty}\frac{5\cdot 2^n-3\cdot 5^{n+1}}{100\cdot 2^n+2\cdot 5^n}=$$

$$\lim_{n\to\infty}\frac{5\cdot 2^n-3\cdot 5^n\cdot 5}{100\cdot 2^n+2\cdot 5^n}=$$

$$\lim_{n\to\infty}\frac{5^n\cdot(5\cdot\frac{2^n}{5^n}-15)}{5^n(100\cdot\frac{2^n}{5^n}+2)}=$$

$$\lim_{n\to\infty}\frac{\cancel{5^n}\cdot(5\cdot\left(\frac{2}{5}\right)^n-15)}{\cancel{5^n}(100\cdot\left(\frac{2}{5}\right)^n+2)}=\lim_{n\to\infty}\frac{5\cdot\overset{\text{ovo je } 0}{\left(\frac{2}{5}\right)^n}-15}{100\cdot\underset{\text{ovo je } 0}{\left(\frac{2}{5}\right)^n}+2}=\frac{-15}{2}$$

3. Odrediti granične vrednosti sledećih nizova:

a) $\lim\limits_{n\to\infty}\sqrt[n]{2^n+3^n+4^n}$

b) $\lim\limits_{n\to\infty}[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+...+\frac{1}{\sqrt{n^2+n}}]$

Rešenje:

Na ovim primerima ćemo upotrebiti takozvanu teoremu o sendviču:

Ako za nizove $(a_n),(b_n),(c_n)$, važi $a_n\le b_n\le c_n$ i ako je $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}c_n=A$ onda je i $\lim\limits_{n\to\infty}b_n=A$

a)

$$\lim_{n\to\infty}\sqrt[n]{2^n+3^n+4^n}=?$$

Primetimo da je :

$$\sqrt[n]{2^n+3^n+4^n}\le\sqrt[n]{4^n+4^n+4^n}=\sqrt[n]{3\cdot 4^n}=\sqrt[n]{3}\cdot\sqrt[n]{4^n}=4\cdot\sqrt[n]{3}$$

Kako je $\sqrt[n]{3}=3^{\frac{1}{n}}$, a kad $n\to\infty$ imamo $\sqrt[n]{3}=3^{\frac{1}{n}}=3^{\frac{1}{\infty}}=3^0=1$

$Dakle:\sqrt[n]{2^n+3^n+4^n}\le\sqrt[n]{4^n+4^n+4^n}=\sqrt[n]{3\cdot 4^n}=\sqrt[n]{3}\cdot\sqrt[n]{4^n}=4\cdot\sqrt[n]{3}=4$

Ako stavimo da je n=2 , dobijamo $\sqrt[n]{2^n+3^n+4^n}=\sqrt{2^2+3^2+4^2}=\sqrt{4+9+16}=\sqrt{29}\ge 4$

Po teoremi o sendviču je $4\le\sqrt[n]{2^n+3^n+4^n}\le 4\to\lim\limits_{n\to\infty}\sqrt[n]{2^n+3^n+4^n}=4$

b)

$$\lim_{n\to\infty}[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+...+\frac{1}{\sqrt{n^2+n}}]=?$$

Razmišljamo ovako:

$$n^2+1\ge n^2 \qquad n^2+2\ge n^2 \qquad n^2+n\ge n^2$$

$$\sqrt{n^2+1}\ge\sqrt{n^2}=n \qquad \sqrt{n^2+2}\ge\sqrt{n^2}=n \qquad \ldots\ldots. \qquad \sqrt{n^2+n}\ge\sqrt{n^2}=n$$

$$\frac{1}{\sqrt{n^2+1}}\le\frac{1}{n} \qquad \frac{1}{\sqrt{n^2+2}}\le\frac{1}{n} \qquad \frac{1}{\sqrt{n^2+n}}\le\frac{1}{n}$$

Ako saberemo sve ove nejednakosti :

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+...+\frac{1}{\sqrt{n^2+n}}\le\frac{1}{n}+\frac{1}{n}+.......+\frac{1}{n}=n\cdot\frac{1}{n}=1$$

Dalje moramo dokazati da je $\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+...+\frac{1}{\sqrt{n^2+n}}\ge 1$

Dokazaćemo da je

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+...+\frac{1}{\sqrt{n^2+n}}\ge\frac{n}{\sqrt{n^2+n}},$$

a znamo da je $\lim\limits_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\frac{n}{\sqrt{n^2(1+\frac{1}{n})}}=\lim\limits_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\cdot\sqrt{(1+\frac{1}{n})}}=1$

Razmišljamo ovako:

$$n^2+n \geq n^2+1 \qquad n^2+n \geq n^2+2 \qquad n^2+n \geq n^2+n$$

$$\sqrt{n^2+n} \geq \sqrt{n^2+1} \qquad \sqrt{n^2+n} \geq \sqrt{n^2+2} \qquad \sqrt{n^2+n} \geq \sqrt{n^2+n}$$

$$\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} \qquad \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+2}} \qquad \ldots\ldots\ldots \qquad \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+n}}$$

$$\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n}} \qquad \frac{1}{\sqrt{n^2+2}} \geq \frac{1}{\sqrt{n^2+n}} \qquad \frac{1}{\sqrt{n^2+n}} \geq \frac{1}{\sqrt{n^2+n}}$$

Ako saberemo sve ove nejednakosti : $\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\ldots+\frac{1}{\sqrt{n^2+n}} \geq \frac{n}{\sqrt{n^2+n}}$

Dakle: $1 \leq \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\ldots+\frac{1}{\sqrt{n^2+n}} \leq 1$ pa je

$$\lim_{n\to\infty}[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\ldots+\frac{1}{\sqrt{n^2+n}}]=1$$