Transformacija zbira i razlike trigonometrijskih funkcija u proizvod i obrnuto

Formule su:

1. $\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
2. $\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
3. $\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
4. $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

5. $tg\alpha\pm tg\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$
6. $ctg\alpha\pm ctg\beta=\frac{\sin(\alpha\pm\beta)}{\sin\alpha\sin\beta}$

7. $\sin x\cdot\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$
8. $\cos x\cdot\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$
9. $\cos x\cdot\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$
10. $\sin x\cdot\sin y=-\frac{1}{2}[\cos(x+y)-\cos(x-y)]$

<u>Primeri :</u>

1) Transformisati u proizvod

a) $\sin 20^o+\cos 50^o$

b) $\sin 56^o-\cos 56^o$

v) $\sin\alpha-\sin\beta$

a) $\sin 20^o+\cos 50^o=$(pošto nam formula za zbir sinusa i kosinusa, upotrebom veza u I kvandrantu, prebacićemo: $\cos 50^o=\sin 40^o$)

$$
\begin{aligned}
&=\sin 20^o+\sin 40^o=2\sin\frac{20^o+40^o}{2}\cos\frac{20^o-40^o}{2}\\
&=2\sin 30^o\cos(-10^o)\\
&=2\sin 30^o\cos 10^o\\
&=2\cdot\frac{1}{2}\cos 10^o=\cos 10^o
\end{aligned}
$$

b)

$$\sin 56^o - \cos 56^o =$$
$$= \sin 56^o - \sin 34^o$$
$$= 2\cos\frac{56^o + 34^o}{2}\sin\frac{56^o - 34^o}{2}$$
$$= 2\cos 45^o \sin 11^o$$
$$= 2\cdot\frac{\sqrt{2}}{2}\sin 11^o = \sqrt{2}\sin 11^o$$

v)

$$\sin\alpha - \sin\beta =$$
$$= \sin\alpha - \sin\left(\frac{\pi}{2} - \alpha\right)$$
$$= 2\cos\frac{\cancel{\alpha} + \frac{\pi}{2} \cancel{-\alpha}}{2}\sin\frac{\alpha - \left(\frac{\pi}{2} - \alpha\right)}{2}$$
$$= 2\cos\frac{\pi}{4}\sin\frac{\alpha - \frac{\pi}{2} + \alpha}{2}$$
$$= 2\cos\frac{\pi}{4}\sin\frac{2\alpha - \frac{\pi}{2}}{2}$$
$$= 2\cdot\frac{\sqrt{2}}{2}\sin\left(\alpha - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$$

2) Dokazati da je:

a) $\sin 15^o \sin 75^o = 0,25$

b) $\cos 135^o \cos 45^o = -0,5$

a)

$$\sin 15^o \sin 75^o = \frac{1}{2}\left[\sin(15^o + 75^o) + \sin(15^o - 75^o)\right]$$
$$= \frac{1}{2}\left[\sin 90^o + \sin(-60^o)\right]$$ pazi: sinx je neparna funkcija sin(-x) = -sinx
$$= \frac{1}{2}\left[\sin 90^o - \sin 60^o\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{2}\right] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = 0,25$$

b)

$$\cos 135^o \cos 45^o = \frac{1}{2}\left[\cos(135^o - 45^o) + \cos(135^o + 45^o)\right]$$

$$= \frac{1}{2}\left[\cos 90^o + \cos 180^o\right]$$

$$= \frac{1}{2}[0-1] = -\frac{1}{2} = -0,5$$

3) Izračunati

a) sin5x sin3x=?

b) $\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4} = ?$

a)

$$\sin 5x \sin 3x = \frac{1}{2}[\cos(5x-3x) - \cos(5x+3x)]$$

$$= \frac{1}{2}[\cos 2x - \cos 8x]$$

b)

$\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4} =$(grupišemo prva dva na koja ćemo upotrebiti formulu, a $\cos\frac{x}{4}$ neka sačeka!)

$$= \left(\cos\frac{x}{2}\cos\frac{x}{3}\right)\cdot\cos\frac{x}{4}$$

$$= \frac{1}{2}\left[\cos\left(\frac{x}{2}-\frac{x}{3}\right) + \cos\left(\frac{x}{2}+\frac{x}{3}\right)\right]\cdot\cos\frac{x}{4}$$

$$= \frac{1}{2}\left[\cos\left(\frac{x}{6}\right) + \cos\left(\frac{5x}{6}\right)\right]\cdot\cos\frac{x}{4}$$

$= \frac{1}{2}(\cos\frac{x}{6}\cdot\cos\frac{x}{4}) + \frac{1}{2}(\cos\frac{5x}{6}\cdot\cos\frac{x}{4}) \rightarrow$ ovde opet upotrebimo formulu za izraze u zagradama

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\cos\left(\frac{x}{6}-\frac{x}{4}\right) + \cos\left(\frac{x}{6}+\frac{x}{4}\right)\right] + \frac{1}{2}\cdot\frac{1}{2}\left[\cos\left(\frac{5x}{6}-\frac{x}{4}\right) + \cos\left(\frac{5x}{6}+\frac{x}{4}\right)\right]$$

$$= \frac{1}{4}\left[\cos\frac{-x}{12} + \cos\frac{5x}{12}\right] + \frac{1}{4}\left[\cos\frac{7x}{12} + \cos\frac{13x}{12}\right]$$

$$= \frac{1}{4}\left[\cos\frac{x}{12} + \cos\frac{5x}{12} + \cos\frac{7x}{12} + \cos\frac{13x}{12}\right]$$

4) Dokazati da je : **a)** $\sin 20^o \cdot \sin 40^o \cdot \sin 80^o = \frac{\sqrt{3}}{8}$

b) $\cos 10^o \cos 50^o \cdot \cos 70^o = \frac{\sqrt{3}}{8}$

Rešenje:

a) $\sin 20^o \cdot \sin 40^o \cdot \sin 80^o =$ upakujemo prvi i treći činilac po formuli.

$$\sin 20^o \cdot \sin 40^o \cdot \sin 80^o = (\sin 20^o \cdot \sin 80^o) \cdot \sin 40^o = \frac{1}{2}[\cos(20^0 - 80^0) - \cos(20^0 + 80^0)] \cdot \sin 40^o$$

$$= \frac{1}{2}\left[\cos 60^o - \cos 100^o\right]\sin 40^o$$

$$= \frac{1}{2}\sin 40^o\left[\frac{1}{2} - \cos 100^o\right] \rightarrow \{\text{Znamo da je } -\cos 100^o = \cos 80^o, \text{pa je }\} = \frac{1}{2}\sin 40^o\left[\frac{1}{2} + \cos 80^0\right]$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{2}\sin 40^o \cos 80^o$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{2}\left[\frac{1}{2}(\sin 120^o + \sin(-40^o))\right]$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{4}(\sin 120^o - \sin 40^o)$$

$$= \cancel{\frac{1}{4}\sin 40^o} + \frac{1}{4}\sin 120^o \cancel{- \frac{1}{4}\sin 40^o}$$

$$= \frac{1}{4}\sin 120^o = \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$$

b)

Pa ovo je ustvari isti zadatak!

Zašto?

$$\cos 10^o = \sin 80^o$$
$$\cos 50^o = \sin 40^o$$
$$\cos 70^o = \sin 20^o$$

Dakle $\cos 10^o \cdot \cos 50^o \cdot \cos 70^o$ je isto $\frac{\sqrt{3}}{8}$

5) Transformisati u proizvod $\sin x+\sin y+\sin z$, **ako je** $x+y+z=\pi$

Rešenje:

$\sin x+\sin y+\sin z=$

$\sin x+\sin y+\sin\left[\pi-(x+y)\right]=$

$\sin x+\sin y+\sin(x+y)$ $\rightarrow$ Upotrebimo $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ i $\sin\alpha=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$2\sin\frac{x+y}{2}\cos\frac{x-y}{2}+2\sin\frac{x+y}{2}\cos\frac{x+y}{2}=$ Izvučemo zajednički $2\sin\frac{x+y}{2}$

$2\sin\frac{x+y}{2}[\cos\frac{x-y}{2}+\cos\frac{x+y}{2}]=$ Sad formula za $\cos\alpha+\cos\beta$

$2\sin\frac{x+y}{2}\cdot 2\cdot\cos\frac{x}{2}\cdot\cos\frac{y}{2}=$

$4\sin\frac{x+y}{2}\cos\frac{x}{2}\cos\frac{y}{2}=$

transformišemo $\sin\frac{x+y}{2}$

$\sin\frac{x+y}{2}=\sin\frac{\pi-z}{2}=\sin\left(\frac{\pi}{2}-\frac{z}{2}\right)=\sin\left(90^o-\frac{z}{2}\right)=\cos\frac{z}{2}$ po formuli za veze u I kvandrantu.

Dakle: $\sin x+\sin y+\sin z=4\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}$ **Simpatično, zar ne?**

6) Dokazati da je:

$$\sin 495^o-\sin 795^o+\sin 1095^o=0$$

Rešenje:

Najpre ćemo ove uglove prebaciti u I kvadrant, da nam bude lakše!

$\sin 495^o=\sin(495^o-360^o)=\sin 135^o=\cos 45^o$

$\sin 795^o=\sin(795-2\cdot 360^o)=\sin 75^o=\cos 15^o$

$\sin 1095^o=\sin(1095^o-3\cdot 360^o)=\sin 15^o$

Znači sada imamo:

$\cos 45^o-\cos 15^o+\sin 15^o=$ na prva dva člana upotrebimo formulu...

$-2\sin\frac{45^o+15^o}{2}\sin\frac{45^o-15^o}{2}+\sin 15^o=$

$-2\sin 30^o\sin 15^o+\sin 15^o=$

$-2\frac{1}{2}\sin 15^o+\sin 15^o=-\sin 15^o+\sin 15^o=0$

7) Dokazati da je:

$$tg9^o - tg27^o - tg63^o + tg81^o = 4$$

Rešenje:

Pregrupišemo prvo članove:

$$\left(tg81^o + tg9^o\right) - \left(tg63^o + tg27^o\right) = imamo\ \ formule$$

$$\frac{\sin(81^o + 9^o)}{\cos 81^o \cos 9^o} - \frac{\sin(63^o + 27^o)}{\cos 63^o \cos 27^o} =$$

$$\frac{\sin 90^o}{\cos 81^o \cos 9^o} - \frac{\sin 90^o}{\cos 63^o \cos 27^o} = (\sin 90^o = 1)$$

$$\frac{1}{\cos 81^o \cos 9^o} - \frac{1}{\cos 63^o \cos 27^o} = \begin{pmatrix} \cos 81^o = \sin 9^o \\ \cos 63^o = \sin 27^o \end{pmatrix}$$

$$\frac{1}{\sin 9^o \cos 9^o} - \frac{1}{\sin 27^o \cos 27^o} = \left(dodamo\ \frac{2}{2}\right)$$

$$\frac{2}{2\sin 9^o \cos 9^o} - \frac{2}{2\sin 27^o \cos 27^o} = (\sin 2\alpha = 2\sin\alpha\cos\alpha)$$

$$\frac{2}{\sin 18^o} - \frac{2}{\sin 54^o} = (zajednicki)$$

$$\frac{2(\sin 54^o - \sin 18^o)}{\sin 18^o \sin 54^o} = (gore\ \ formula)$$

$$\frac{2\cdot 2\cos\frac{54^o + 18^o}{2}\sin\frac{54^o - 18^o}{2}}{\sin 18^o \sin 54^o} = \frac{4\cos 36^o \cancel{\sin 18^o}}{\cancel{\sin 18^o}\sin 54^o} = \frac{4\cos 36^o}{\sin 54^o} = \frac{4\cancel{\cos 36^o}}{\cancel{\cos 36^o}} = \boxed{4}$$

5) Izračunati $\sin 36^o$ **bez upotrebe tablica.**

Rešenje:

Znamo da važi veza u I kvandrantu:

$$\sin 36^o = \cos 54^o$$

odnosno $\quad \sin 2\cdot 18^o = \cos 3\cdot 18^o$

formula za $\sin 2\alpha$ imamo: $\sin 2\alpha = 2\sin\alpha\cos\alpha$ a formula za $\cos 3\alpha$ smo izveli (pogledaj) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$. Upotrebimo ih:

$$\sin 2\cdot 18^o = 2\sin 18^o \cos 18^o$$

$$\cos 3\cdot 18^o = 4\cos^3 18^o - 3\cos 18^o$$

Pa je:

$$4\cos^3 18^o - 3\cos 18^o = 2\sin 18^o \cos 18^o$$

(sve podelimo sa $\cos 18^o$)

$$4\cos^2 18^o - 3 = 2\sin 18^o$$

(onda je $\sin^2 18^0 + \cos^2 18^0 = 1 \Rightarrow \cos^2 18^o = 1 - \sin^2 18^o$)

$$4(1-\sin^2 18) - 3 - 2\sin 18^o = 0$$
$$4 - 4\sin^2 18 - 3 - 2\sin 18^o = 0$$
$$4\sin^2 18 + 2\sin 18^o - 1 = 0$$

(uzmimo smenu $\sin 18^o = t$)

$$4t^2 + 2t - 1 = 0$$
$$t_{1,2} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{2(-1 \pm \sqrt{5})}{8}$$
$$t_{1,2} = \frac{-1 \pm \sqrt{5}}{4}$$
$$t_1 = \frac{-1 + \sqrt{5}}{4}$$
$$t_2 = \frac{-1 - \sqrt{5}}{4}$$
$$\sin 18^o = \frac{\sqrt{5} - 1}{4}$$

Nadjimo sad $\cos 18^o$

$$\sin^2 18^o + \cos^2 18^o = 1$$
$$\cos^2 18^o = 1 - \left(\frac{\sqrt{5} - 1}{4} \right)^2$$
$$\cos^2 18^o = 1 - \frac{5 - 2\sqrt{5} + 1}{16}$$
$$\cos^2 18^o = \frac{16 - 6 + 2\sqrt{5}}{16}$$
$$\cos^2 18^o = \frac{10 + 2\sqrt{5}}{16}$$
$$\cos 18^o = \sqrt{\frac{10 + 2\sqrt{5}}{16}}$$
$$\cos 18^o = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Zamenimo dobijena rešenja i malo prisredimo:

$$\sin 36^o = 2\sin 18^o \cos 18^o$$

$$\sin 36^o = 2\cdot\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$(\textit{ubacimo } \sqrt{5}-1 \textit{ pod koren})$

$$\sin 36^o = \frac{\sqrt{\left(\sqrt{5}-1\right)^2\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{\left(5-2\sqrt{5}+1\right)\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{\left(6-2\sqrt{5}\right)\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{60+12\sqrt{5}-20\sqrt{5}-20}}{8}$$

$$\sin 36^o = \frac{\sqrt{40-8\sqrt{5}}}{8}$$

$$\sin 36^o = \frac{\sqrt{8\left(5-\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{2\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

$$\boxed{\sin 36^o = \frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}}$$