

Transformacija zbir a i razlike trigonometrijskih funkcija u proizvod i obrnuto

Formule su:

$$1. \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$2. \sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$3. \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$4. \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$5. \operatorname{tg} \alpha \pm \operatorname{tg} \beta = \frac{\sin(\alpha \pm \beta)}{\cos \alpha \cos \beta}$$

$$6. \operatorname{ctg} \alpha \pm \operatorname{ctg} \beta = \frac{\sin(\alpha \pm \beta)}{\sin \alpha \sin \beta}$$

$$7. \sin x \cdot \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)]$$

$$8. \cos x \cdot \sin y = \frac{1}{2} [\sin(x + y) - \sin(x - y)]$$

$$9. \cos x \cdot \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]$$

$$10. \sin x \cdot \sin y = -\frac{1}{2} [\cos(x + y) - \cos(x - y)]$$

Primeri :

1) Transformisati u proizvod

a) $\sin 20^\circ + \cos 50^\circ$

b) $\sin 56^\circ - \cos 56^\circ$

v) $\sin \alpha - \sin \beta$ (ako je u pitanju pravougli trougao)

a) $\sin 20^\circ + \cos 50^\circ =$ (pošto nam formula za zbir sinusa i kosinusa, upotrebom veza u I kvadrantu, prebacićemo: $\cos 50^\circ = \sin 40^\circ$)

$$= \sin 20^\circ + \sin 40^\circ = 2 \sin \frac{20^\circ + 40^\circ}{2} \cos \frac{20^\circ - 40^\circ}{2}$$

$$= 2 \sin 30^\circ \cos(-10^\circ)$$

$$= 2 \sin 30^\circ \cos 10^\circ$$

$$= 2 \cdot \frac{1}{2} \cos 10^\circ = \cos 10^\circ$$

b)

$$\begin{aligned}\sin 56^\circ - \cos 56^\circ &= \\ &= \sin 56^\circ - \sin 34^\circ \\ &= 2 \cos \frac{56^\circ + 34^\circ}{2} \sin \frac{56^\circ - 34^\circ}{2} \\ &= 2 \cos 45^\circ \sin 11^\circ \\ &= 2 \cdot \frac{\sqrt{2}}{2} \sin 11^\circ = \sqrt{2} \sin 11^\circ\end{aligned}$$

v)

$$\begin{aligned}\sin \alpha - \sin \beta &= \\ &= \sin \alpha - \sin \left(\frac{\pi}{2} - \alpha \right) \\ &= 2 \cos \frac{\cancel{\alpha} + \frac{\pi}{2} - \cancel{\alpha}}{2} \sin \frac{\alpha - \left(\frac{\pi}{2} - \alpha \right)}{2} \\ &= 2 \cos \frac{\pi}{4} \sin \frac{\alpha - \frac{\pi}{2} + \alpha}{2} \\ &= 2 \cos \frac{\pi}{4} \sin \frac{2\alpha - \frac{\pi}{2}}{2} \\ &= 2 \cdot \frac{\sqrt{2}}{2} \sin \left(\alpha - \frac{\pi}{4} \right) \\ &= \sqrt{2} \sin \left(\alpha - \frac{\pi}{4} \right)\end{aligned}$$

2) Dokazati da je:

a) $\sin 15^\circ \cos 75^\circ = 0,25$

b) $\cos 135^\circ \cos 45^\circ = -0,5$

a)

$$\begin{aligned}\sin 15^\circ \cos 75^\circ &= \frac{1}{2} \left[\sin(15^\circ + 75^\circ) + \sin(15^\circ - 75^\circ) \right] \\ &= \frac{1}{2} \left[\sin 90^\circ + \sin(-60^\circ) \right] \quad \text{pazi: } \sin x \text{ je neparna funkcija } \sin(-x) = -\sin x \\ &= \frac{1}{2} \left[\sin 90^\circ - \sin 60^\circ \right] \\ &= \frac{1}{2} \left[1 - \frac{1}{2} \right] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0,25\end{aligned}$$

b)

$$\begin{aligned}\cos 135^\circ \cos 45^\circ &= \frac{1}{2} [\cos(135^\circ - 45^\circ) + \cos(135^\circ + 45^\circ)] \\ &= \frac{1}{2} [\cos 90^\circ + \cos 180^\circ] \\ &= \frac{1}{2} [0 - 1] = -\frac{1}{2} = -0,5\end{aligned}$$

3) Izračunati

a) $\sin 5x \sin 3x = ?$

b) $\cos \frac{x}{2} \cos \frac{x}{3} \cos \frac{x}{4} = ?$

a)

$$\begin{aligned}\sin 5x \sin 3x &= \frac{1}{2} [\cos(5x - 3x) - \cos(5x + 3x)] \\ &= \frac{1}{2} [\cos 2x - \cos 8x]\end{aligned}$$

b)

$\cos \frac{x}{2} \cos \frac{x}{3} \cos \frac{x}{4} =$ (grupiřemo prva dva na koja ćemo upotrebiti formulu, a $\cos \frac{x}{4}$ neka saćeka!)

$$\begin{aligned}&= \left(\cos \frac{x}{2} \cos \frac{x}{3} \right) \cdot \cos \frac{x}{4} \\ &= \frac{1}{2} \left[\cos \left(\frac{x}{2} - \frac{x}{3} \right) + \cos \left(\frac{x}{2} + \frac{x}{3} \right) \right] \cdot \cos \frac{x}{4} \\ &= \frac{1}{2} \left[\cos \left(\frac{x}{6} \right) + \cos \left(\frac{5x}{6} \right) \right] \cdot \cos \frac{x}{4} \\ &= \frac{1}{2} \left(\cos \frac{x}{6} \cdot \cos \frac{x}{4} \right) + \frac{1}{2} \left(\cos \frac{5x}{6} \cdot \cos \frac{x}{4} \right) \rightarrow \text{ovde opet upotrebimo formulu za izraze u zagradama} \\ &= \frac{1}{2} \cdot \frac{1}{2} \left[\cos \left(\frac{x}{6} - \frac{x}{4} \right) + \cos \left(\frac{x}{6} + \frac{x}{4} \right) \right] + \frac{1}{2} \cdot \frac{1}{2} \left[\cos \left(\frac{5x}{6} - \frac{x}{4} \right) + \cos \left(\frac{5x}{6} + \frac{x}{4} \right) \right] \\ &= \frac{1}{4} \left[\cos \frac{-x}{12} + \cos \frac{5x}{12} \right] + \frac{1}{4} \left[\cos \frac{7x}{12} + \cos \frac{13x}{12} \right] \\ &= \frac{1}{4} \left[\cos \frac{x}{12} + \cos \frac{5x}{12} + \cos \frac{7x}{12} + \cos \frac{13x}{12} \right]\end{aligned}$$

4) Dokazati da je :

a) $\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \frac{\sqrt{3}}{8}$

b) $\cos 10^\circ \cos 50^\circ \cdot \cos 70^\circ = \frac{\sqrt{3}}{8}$

Rešenje:

a) $\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ =$ upakujemo prvi i treći činilac po formuli.

$$\begin{aligned} \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ &= (\sin 20^\circ \cdot \sin 80^\circ) \cdot \sin 40^\circ = \frac{1}{2} [\cos(20^\circ - 80^\circ) - \cos(20^\circ + 80^\circ)] \cdot \sin 40^\circ \\ &= \frac{1}{2} [\cos 60^\circ - \cos 100^\circ] \sin 40^\circ \\ &= \frac{1}{2} \sin 40^\circ \left[\frac{1}{2} - \cos 100^\circ \right] \rightarrow \{ \text{Znamo da je } -\cos 100^\circ = \cos 80^\circ, \text{ pa je } \} = \frac{1}{2} \sin 40^\circ \left[\frac{1}{2} + \cos 80^\circ \right] \\ &= \frac{1}{4} \sin 40^\circ + \frac{1}{2} \sin 40^\circ \cos 80^\circ \\ &= \frac{1}{4} \sin 40^\circ + \frac{1}{2} \left[\frac{1}{2} (\sin 120^\circ + \sin(-40^\circ)) \right] \\ &= \frac{1}{4} \sin 40^\circ + \frac{1}{4} (\sin 120^\circ - \sin 40^\circ) \\ &= \cancel{\frac{1}{4} \sin 40^\circ} + \frac{1}{4} \sin 120^\circ - \cancel{\frac{1}{4} \sin 40^\circ} \\ &= \frac{1}{4} \sin 120^\circ = \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} \end{aligned}$$

b)

Pa ovo je ustvari isti zadatak!

Zašto?

$$\cos 10^\circ = \sin 80^\circ$$

$$\cos 50^\circ = \sin 40^\circ$$

$$\cos 70^\circ = \sin 20^\circ$$

Dakle $\cos 10^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ$ je isto $\frac{\sqrt{3}}{8}$

5) Transformisati u proizvod $\sin x + \sin y + \sin z$, ako je $x + y + z = \pi$

Rešenje:

$$\sin x + \sin y + \sin z =$$

$$\sin x + \sin y + \sin[\pi - (x + y)] =$$

$$\sin x + \sin y + \sin(x + y) \rightarrow \text{Upotrebimo } \sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \text{ i } \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

$$2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} + 2 \sin \frac{x+y}{2} \cos \frac{x+y}{2} = \text{ Izvučemo zajednički } 2 \sin \frac{x+y}{2}$$

$$2 \sin \frac{x+y}{2} [\cos \frac{x-y}{2} + \cos \frac{x+y}{2}] = \text{ Sad formula za } \cos \alpha + \cos \beta$$

$$2 \sin \frac{x+y}{2} \cdot 2 \cdot \cos \frac{x}{2} \cdot \cos \frac{y}{2} =$$

$$4 \sin \frac{x+y}{2} \cos \frac{x}{2} \cos \frac{y}{2} =$$

$$\text{transformišemo } \sin \frac{x+y}{2}$$

$$\sin \frac{x+y}{2} = \sin \frac{\pi - z}{2} = \sin \left(\frac{\pi}{2} - \frac{z}{2} \right) = \sin \left(90^\circ - \frac{z}{2} \right) = \cos \frac{z}{2} \text{ po formuli za veze u I}$$

kvadrantu.

$$\text{Dakle: } \sin x + \sin y + \sin z = 4 \cos \frac{x}{2} \cos \frac{y}{2} \cos \frac{z}{2} \quad \text{Simpatično, zar ne?}$$

6) Dokazati da je:

$$\sin 495^\circ - \sin 795^\circ + \sin 1095^\circ = 0$$

Rešenje:

Najpre ćemo ove uglove prebaciti u I kvadrant, da nam bude lakše!

$$\sin 495^\circ = \sin(495^\circ - 360^\circ) = \sin 135^\circ = \cos 45^\circ$$

$$\sin 795^\circ = \sin(795 - 2 \cdot 360^\circ) = \sin 75^\circ = \cos 15^\circ$$

$$\sin 1095^\circ = \sin(1095^\circ - 3 \cdot 360^\circ) = \sin 15^\circ$$

Znači sada imamo:

$$\cos 45^\circ - \cos 15^\circ + \sin 15^\circ = \text{ na prva dva člana upotrebimo formulu...}$$

$$-2 \sin \frac{45^\circ + 15^\circ}{2} \sin \frac{45^\circ - 15^\circ}{2} + \sin 15^\circ =$$

$$-2 \sin 30^\circ \sin 15^\circ + \sin 15^\circ =$$

$$-2 \frac{1}{2} \sin 15^\circ + \sin 15^\circ = -\sin 15^\circ + \sin 15^\circ = 0$$

7) Dokazati da je:

$$\operatorname{tg} 9^{\circ} - \operatorname{tg} 27^{\circ} - \operatorname{tg} 63^{\circ} + \operatorname{tg} 81^{\circ} = 4$$

Rešenje:

Pregrupišemo prvo članove:

$$\begin{aligned} & (\operatorname{tg} 81^{\circ} + \operatorname{tg} 9^{\circ}) - (\operatorname{tg} 63^{\circ} + \operatorname{tg} 27^{\circ}) = \text{imamo formule} \\ & \frac{\sin(81^{\circ} + 9^{\circ})}{\cos 81^{\circ} \cos 9^{\circ}} - \frac{\sin(63^{\circ} + 27^{\circ})}{\cos 63^{\circ} \cos 27^{\circ}} = \\ & \frac{\sin 90^{\circ}}{\cos 81^{\circ} \cos 9^{\circ}} - \frac{\sin 90^{\circ}}{\cos 63^{\circ} \cos 27^{\circ}} = (\sin 90^{\circ} = 1) \\ & \frac{1}{\cos 81^{\circ} \cos 9^{\circ}} - \frac{1}{\cos 63^{\circ} \cos 27^{\circ}} = \begin{pmatrix} \cos 81^{\circ} = \sin 9^{\circ} \\ \cos 63^{\circ} = \sin 27^{\circ} \end{pmatrix} \\ & \frac{1}{\sin 9^{\circ} \cos 9^{\circ}} - \frac{1}{\sin 27^{\circ} \cos 27^{\circ}} = \left(\text{dodamo } \frac{2}{2} \right) \\ & \frac{2}{2 \sin 9^{\circ} \cos 9^{\circ}} - \frac{2}{2 \sin 27^{\circ} \cos 27^{\circ}} = (\sin 2\alpha = 2 \sin \alpha \cos \alpha) \\ & \frac{2}{\sin 18^{\circ}} - \frac{2}{\sin 54^{\circ}} = (\text{zajednicki}) \\ & \frac{2(\sin 54^{\circ} - \sin 18^{\circ})}{\sin 18^{\circ} \sin 54^{\circ}} = (\text{gore formula}) \\ & \frac{2 \cdot 2 \cos \frac{54^{\circ} + 18^{\circ}}{2} \sin \frac{54^{\circ} - 18^{\circ}}{2}}{\sin 18^{\circ} \sin 54^{\circ}} = \frac{4 \cos 36^{\circ} \cancel{\sin 18^{\circ}}}{\cancel{\sin 18^{\circ}} \sin 54^{\circ}} = \frac{4 \cos 36^{\circ}}{\sin 54^{\circ}} = \frac{4 \cancel{\cos 36^{\circ}}}{\cancel{\cos 36^{\circ}}} = \boxed{4} \end{aligned}$$

5) Izračunati $\sin 36^{\circ}$ bez upotrebe tablica.

Rešenje:

Znamo da važi veza u I kvadrantu:

$$\sin 36^{\circ} = \cos 54^{\circ}$$

$$\text{odnosno} \quad \sin 2 \cdot 18^{\circ} = \cos 3 \cdot 18^{\circ}$$

formula za $\sin 2\alpha$ imamo: $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ a formula za $\cos 3\alpha$ smo izveli (pogledaj) $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$. Upotrebimo ih:

$$\sin 2 \cdot 18^{\circ} = 2 \sin 18^{\circ} \cos 18^{\circ}$$

$$\cos 3 \cdot 18^{\circ} = 4 \cos^3 18^{\circ} - 3 \cos 18^{\circ}$$

Pa je:

$$4\cos^3 18^\circ - 3\cos 18^\circ = 2\sin 18^\circ \cos 18^\circ$$

(sve podelimo sa $\cos 18^\circ$)

$$4\cos^2 18^\circ - 3 = 2\sin 18^\circ$$

(onda je $\sin^2 18^\circ + \cos^2 18^\circ = 1 \Rightarrow \cos^2 18^\circ = 1 - \sin^2 18^\circ$)

$$4(1 - \sin^2 18^\circ) - 3 - 2\sin 18^\circ = 0$$

$$4 - 4\sin^2 18^\circ - 3 - 2\sin 18^\circ = 0$$

$$4\sin^2 18^\circ + 2\sin 18^\circ - 1 = 0$$

(uzmimo smenu $\sin 18^\circ = t$)

$$4t^2 + 2t - 1 = 0$$

$$t_{1,2} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{2(-1 \pm \sqrt{5})}{8}$$

$$t_{1,2} = \frac{-1 \pm \sqrt{5}}{4}$$

$$t_1 = \frac{-1 + \sqrt{5}}{4}$$

$$t_2 = \frac{-1 - \sqrt{5}}{4}$$

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

Nadjimo sad $\cos 18^\circ$

$$\sin^2 18^\circ + \cos^2 18^\circ = 1$$

$$\cos^2 18^\circ = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$\cos^2 18^\circ = 1 - \frac{5 - 2\sqrt{5} + 1}{16}$$

$$\cos^2 18^\circ = \frac{16 - 6 + 2\sqrt{5}}{16}$$

$$\cos^2 18^\circ = \frac{10 + 2\sqrt{5}}{16}$$

$$\cos 18^\circ = \sqrt{\frac{10 + 2\sqrt{5}}{16}}$$

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Zamenimo dobijena rešenja i malo prisredimo:

$$\sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ$$

$$\sin 36^\circ = 2 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(ubacimo $\sqrt{5}-1$ pod koren)

$$\sin 36^\circ = \frac{\sqrt{(\sqrt{5}-1)^2(10+2\sqrt{5})}}{8}$$

$$\sin 36^\circ = \frac{\sqrt{(5-2\sqrt{5}+1)(10+2\sqrt{5})}}{8}$$

$$\sin 36^\circ = \frac{\sqrt{(6-2\sqrt{5})(10+2\sqrt{5})}}{8}$$

$$\sin 36^\circ = \frac{\sqrt{60+12\sqrt{5}-20\sqrt{5}-20}}{8}$$

$$\sin 36^\circ = \frac{\sqrt{40-8\sqrt{5}}}{8}$$

$$\sin 36^\circ = \frac{\sqrt{8(5-\sqrt{5})}}{8}$$

$$\sin 36^\circ = \frac{2\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

$$\boxed{\sin 36^\circ = \frac{\sqrt{2} \cdot \sqrt{5-\sqrt{5}}}{4}}$$

www.matematiranje.in.rs