TRIGONOMETRIJSKE FUNKCIJE POLUUGLOVA

Formule su:

1. $\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$ ili $2\sin^2\frac{\alpha}{2}=1-\cos\alpha$
2. $\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$ ili $2\cos^2\frac{\alpha}{2}=1+\cos\alpha$
3. $tg\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$
4. $ctg\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

Primeri:

1) Odrediti $\cos\frac{\alpha}{2}$, ako je $\sin\alpha=\frac{4}{5}$ i $\alpha\in\left(-\frac{3\pi}{2},-\pi\right)$

Pošto je $\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$ moramo naći $\cos\alpha$.

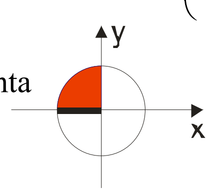
$$\sin^2\alpha+\cos^2\alpha=1$$
$$\cos^2\alpha=1-\sin^2\alpha$$
$$\cos^2\alpha=1-\left(\frac{4}{5}\right)^2$$
$$\cos^2\alpha=1-\frac{16}{25}$$
$$\cos^2\alpha=\frac{9}{25}$$
$$\cos^2\alpha=\pm\sqrt{\frac{9}{25}}$$
$$\cos\alpha=\pm\frac{3}{5}$$

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\cos\frac{\alpha}{2}=-\sqrt{\frac{1-\frac{3}{5}}{2}}$$
$$\cos\frac{\alpha}{2}=-\sqrt{\frac{\frac{2}{5}}{2}}$$
$$\cos\frac{\alpha}{2}=-\frac{1}{\sqrt{5}}$$ racionališemo
$$\cos\frac{\alpha}{2}=-\frac{\sqrt{5}}{5}$$

Pošto je $\alpha\in\left(-\frac{3\pi}{2},-\pi\right)$

iz $\alpha\in\left(-\frac{3\pi}{2},-\pi\right)\Rightarrow\frac{\alpha}{2}\in\left(-\frac{3\pi}{4},\frac{\pi}{2}\right)$

To nam govori da je iz II kvadranta



$$\cos\alpha=-\frac{3}{5}$$

2) Odrediti $\cos\frac{\alpha}{2}$, ako je $\sin\alpha = -\frac{4\sqrt{2}}{9}$ i $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$

Moramo najpre naći $\cos\alpha$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$\cos^2\alpha = 1 - \left(-\frac{4\sqrt{2}}{9}\right)^2$$

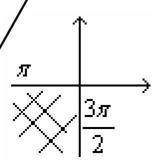
$$\cos^2\alpha = 1 - \frac{16\cdot 2}{81}$$

$$\cos^2\alpha = 1 - \frac{32}{81}$$

$$\cos^2\alpha = \frac{49}{81}$$

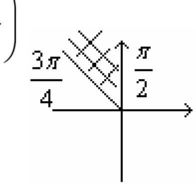
$$\cos\alpha = \pm\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow$$



$$\cos\alpha = -\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \to \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$



Za $\sin\frac{\alpha}{2}$ uzimamo +

$$\sin\frac{\alpha}{2} = +\sqrt{\frac{1-\cos\alpha}{2}} = +\sqrt{\frac{1+\frac{7}{9}}{2}}$$

$$\sin\frac{\alpha}{2} = +\frac{4}{3\sqrt{2}}$$

$$\sin\frac{\alpha}{2} = \frac{4}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{3\cdot 2}$$

$$\sin\frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$$

3) Dokazati:

a) $tg\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ za $\alpha \neq \pi(2k+1), k \in z$

b) $tg\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$ za $\alpha \neq \pi(2k+1), k \in z$

a) Podjimo sad od desne strane da dokažemo levu. $\left[\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]$

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = tg\frac{\alpha}{2}$$

b) $$\frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = tg\frac{\alpha}{2}$$

Odakle $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$? Pa iz: $\sin 2\alpha = 2\sin\alpha\cos\alpha$

4) Ako je $tg\frac{x}{2}=t$ izračunati $\sin x$, $\cos x$ i tgx "preko" *t*,

__Rešenja:__ (ovo će nam biti smena kod trigonometrijskih integrala, zato obratiti pažnju!!!)

$$\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}$$

$$=\frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1}=\frac{2t}{t^2+1}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}$$

$$=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$tgx=\frac{\sin x}{\cos x}=\frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}=\frac{2t}{1-t^2}$$

$$ctgx=\frac{1}{tgx}=\frac{1}{\frac{2t}{1-t^2}}=\frac{1-t^2}{2t}$$

5) Izračunati vrednost izraza $A = \dfrac{\sin x + 2\cos x}{tgx - ctgx}$**, ako je** $tg\dfrac{x}{2} = 2$

Rešenje:

Iskoristićemo $tg\dfrac{x}{2} = 2$, da nadjemo $\cos x$

$$tg\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos\alpha}}$$

$$\sqrt{\frac{1-\cos x}{1+\cos\alpha}} = 2$$

$$\frac{1-\cos x}{1+\cos\alpha} = 4$$

$$1-\cos x = 4(1+\cos)$$

$$1-\cos x = 4+4\cos x$$

$$-\cos x - 4\cos x = 4-1$$

$$-5\cos x = 3$$

$$\cos x = -\frac{3}{5}$$

$$\sin^2 x = 1-\cos^2 x$$

$$\sin^2 x = 1-\frac{9}{25}$$

$$\sin^2 x = \frac{16}{25}$$

$$\sin x = \pm\frac{4}{5}$$

$$\sin x = +\frac{4}{5}$$

$$tgx = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

$$ctgx = -\frac{3}{4}$$

II način

Mogli smo da iskoristimo rezultat prethodnog zadatka:

$$tg\frac{x}{2} = t = 2$$

$$\sin x = \frac{2t}{1+t^2} = \frac{2\cdot 2}{1+2^2} = \frac{4}{5}$$

$$\cos x = \frac{1-t^2}{1+t^2} = \frac{1-2^2}{1+2^2} = \frac{-3}{5}$$

Isto se dobija!

Izračunajmo vrednost izraza A

$$A = \frac{\sin x + 2\cos x}{tgx - ctgx}$$

$$A = \frac{\frac{4}{5} + 2\cdot\left(-\frac{3}{5}\right)}{-\frac{4}{3}+\frac{3}{4}}$$

$$A = \frac{\frac{-2}{5}}{\frac{-7}{12}}$$

$$A = \frac{24}{35}$$

6) Dokaži indetitete:

a) $\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x}=tg\dfrac{x}{2}$

b) $\dfrac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=tg^2\dfrac{x}{2}$

<u>Rešenje:</u> Naravno, podjemo od leve strane pa transformišemo izraz dok ne dodjemo do desne strane!

a) $\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x}=\dfrac{1-\cos x+\sin x}{1+\cos x+\sin x}$

Ideja je: $\begin{cases}1-\cos x=2\sin^2\dfrac{x}{2}\\ 1+\cos x=2\cos^2\dfrac{x}{2}\\ \sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}\end{cases}$

$$=\frac{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=$$ Izvučemo zajednički gore i dole!

$$=\frac{\not{2}\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{\not{2}\cos\frac{x}{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=tg\frac{x}{2}$$

b)

$$\frac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=\frac{2\sin x-2\sin x\cos x}{2\sin x+2\sin x\cos x}=$$

$$=\frac{2\sin x(1-\cos x)}{2\sin x(1+\cos x)}=$$

$$=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}=tg^2\frac{x}{2}$$

7) Izračunati bez upotrebe računskih pomagala $tg7^o30'$

<u>Ideja</u> nam je da iskoristimo jednakost: $tg\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}$

$$tg7^o30'=tg\frac{15^o}{2}=\frac{\sin 15^o}{1+\cos 15^o}$$

Sada moramo naći $\sin 15^o$ i $\cos 15^o$

$$\sin 15^o=\sin\frac{30^o}{2}=\sqrt{\frac{1-\cos 30^o}{2}}=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}=\sqrt{\frac{2-\sqrt{3}}{4}}$$

$\sin 15^o=\frac{\sqrt{2-\sqrt{3}}}{2}$, ovo je tačno, al je malo komplikovano zbog duplog korena koji bi morali da “uništimo’’ preko Lagranžovog indetiteta (pogledaj to), zato ćemo ići:

$$\sin 15^o=\sin(45^o-30^o)=\sin 45^o\cos 30^o-\cos 45^o\sin 30^o=$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$\cos 15^o=\cos(45^o-30^o)=\cos 45^o\cos 30^o+\sin 45^o\sin 30^o=$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

Sad je:

$$tg7^o30'=\frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{1+\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}}=\frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\frac{4+\sqrt{2}\left(\sqrt{3}+1\right)}{4}}$$

$tg7^o30'=\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}+4}=$ odradimo duplu racionalizaciju (pogledaj to u delu korenovanje)

$$\frac{\sqrt{6}-\sqrt{2}}{\left(\sqrt{6}+\sqrt{2}\right)+4}\cdot\frac{\left(\sqrt{6}+\sqrt{2}\right)-4}{\left(\sqrt{6}+\sqrt{2}\right)-4}=\frac{6-2-4\left(\sqrt{6}-\sqrt{2}\right)}{6+2\sqrt{12}+2-16}=\frac{4-4\left(\sqrt{6}-\sqrt{2}\right)}{4\sqrt{3}-8}=$$

$$\frac{4(1-\left(\sqrt{6}-\sqrt{2}\right))}{4(\sqrt{3}-2)}=\frac{1-\left(\sqrt{6}-\sqrt{2}\right)}{\sqrt{3}-2}\cdot\frac{\sqrt{3}+2}{\sqrt{3}+2}=\frac{\sqrt{3}+2-\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{3}+2\right)}{3-4}$$

$$=\frac{\sqrt{3}+2-\sqrt{18}-2\sqrt{6}+\sqrt{6}+2\sqrt{2}}{-1}=\frac{\sqrt{3}+2-3\sqrt{2}-\sqrt{6}+2\sqrt{2}}{-1}$$

$$=\frac{\sqrt{3}+2-\sqrt{2}-\sqrt{6}}{-1}=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$$

$$\boxed{tg7^o30'=\sqrt{6}-\sqrt{3}+\sqrt{2}-2}$$

8) Izračunati vrednost izraza $\frac{\sin 160^o}{\cos^4 40^o-\sin^4 40^o}$

Rešenje:

$$\frac{\sin 160^o}{\cos^4 40^o-\sin^4 40^o}=(\text{ovo dole je razlika kvadrata})$$

$$\frac{\sin 160^o}{(\cos^2 40^o+\sin^2 40^o)(\cos^2 40^o-\sin^2 40^o)}= \text{ ovde je } \cos^2 40^o+\sin^2 40^o=1$$

$$\frac{\sin 160^o}{\cos^2 40^o-\sin^2 40^o}=(\text{ovo dole je } \cos^2 x-\sin^2 x=\cos 2x\text{ })$$

$$\frac{\sin 160^o}{\cos 80^o}=\frac{2\sin 80^o\,\cancel{\cos 80^o}}{\cancel{\cos 80^o}}=2\sin 80^o$$