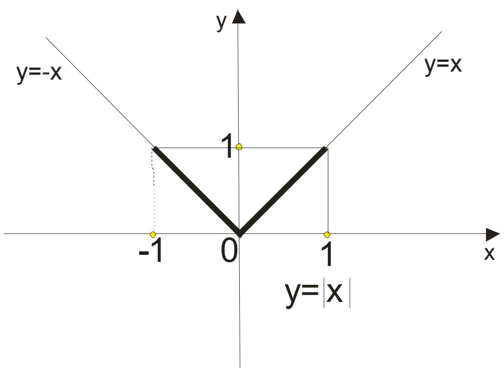
FURIJEOVI REDOVI – ZADACI (I deo)

Primer 1.

Funkciju $y = |x|$ razviti u Furijeov red na intervalu $[-\pi, \pi]$

Rešenje:

Najpre ćemo nacrtati sliku da se podsetimo kako izgleda ova funkcija...



Očigledno je funkcija parna (grafik je simetričan u odnosu na y osu), pa koristimo formule:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \qquad \text{dok je} \qquad b_n = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\cdot(\frac{x^2}{2}) \Big/_0^{\pi} = \frac{2}{\pi}\cdot(\frac{\pi^2}{2}) = \pi$$

Dalje tražimo:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx =$$

Ovaj integral ćemo rešiti uz pomoć parcijalne integracije, izvučimo ga na stranu , bez granica:

$$\int x\cos nx dx = \begin{vmatrix} x = u & \cos nx dx = dv \\ dx = du & \frac{1}{n}\sin nx = v \end{vmatrix} = x\cdot\frac{1}{n}\sin nx - \int \frac{1}{n}\sin nx dx = \frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx dx =$$

$$= \frac{x\sin nx}{n} + \frac{1}{n}\frac{1}{n}\cos nx = \frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx$$

Sad mu stavimo granicu:

$$\left(\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right)\Big/_0^{\pi} = \left(\underbrace{\boxed{\frac{\pi\sin n\pi}{n}}}_{ovo\ je\ 0} + \frac{1}{n^2}\cos n\pi\right) - \left(\frac{0\cdot\sin n\cdot 0}{n} + \frac{1}{n^2}\underbrace{\boxed{\cos n\cdot 0}}_{ovo\ je\ 1}\right) =$$

$$= \frac{1}{n^2}\cos n\pi - \frac{1}{n^2} = \frac{1}{n^2}(\cos n\pi - 1)$$

Onda je :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\cdot\frac{1}{n^2}(\cos n\pi - 1) = \frac{2}{\pi n^2}(\cos n\pi - 1)$$

Naravno da n uzima vrednosti 1,2,3...

Izraz $\cos n\pi$ neizmenično ima vrednosti :

za n=1 je cosπ=-1
za n=2 je cosπ=1
za n=3 je cosπ=-1
za n=4 je cosπ=1
itd.

Dakle , važi da je $\boxed{\cos n\pi = (-1)^n}$

Onda je $a_n = \frac{2}{\pi n^2}((-1)^n - 1)$

Ako je n paran broj , imamo: $a_{2n} = \frac{2}{\pi n^2}((-1)^{2n} - 1) = 0$

Ako je n neparan broj , imamo: $a_{2n-1} = \frac{2}{\pi(2n-1)^2}((-1)^{2n-1} - 1) = \frac{2}{\pi(2n-1)^2}\cdot(-2) = \frac{-4}{\pi(2n-1)^2}$

Vratimo se sada u formulu za razvoj:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{2}\pi+\sum_{n=1}^{\infty}\left(\frac{-4}{\pi(2n-1)^2}\right)\cos(2n-1)x=\frac{1}{2}\pi+\frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

Dakle:

$$\boxed{f(x)=|x|=\frac{1}{2}\pi-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}}$$

Primer 2.

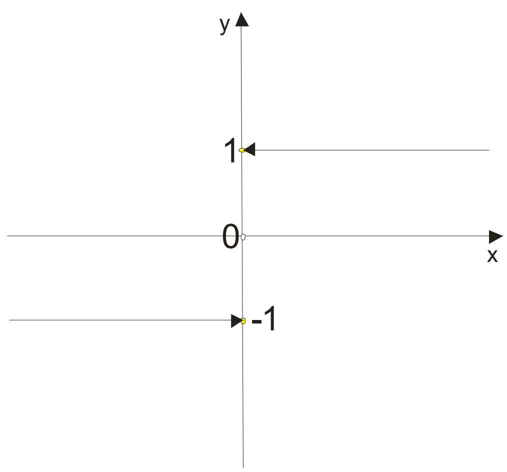
Razviti u Furijeov red funkciju $f(x)=\operatorname{sgn} x$ u intervalu $[-\pi,\pi]$

<u>*Rešenje:*</u>

Najpre malo objašnjenje:

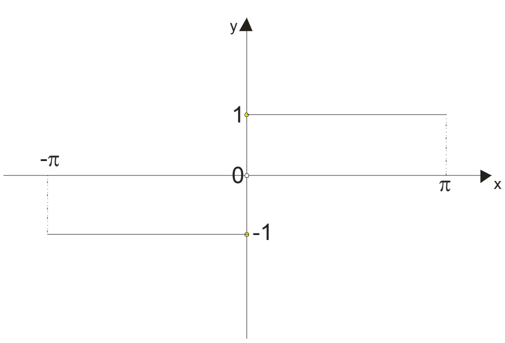
Funkcija *sgnx* se čita signum od x ili po naški znak od x .

Ona je ustvari: $\operatorname{sgn} x=\begin{Bmatrix}-1, & za\ x<0\\ 0, & za\ x=0\\ +1, & za\ x>0\end{Bmatrix}$ pogledajmo sliku:



Ako je $x\neq 0$ onda imamo $\operatorname{sgn} x=\frac{x}{|x|}$

Nama ova funkcija treba na intervalu $[-\pi,\pi]$:



Očigledno je data funkcija neparna, pa koristimo formule: $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$ $\qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right) / {}_0^{\pi} = \frac{2}{\pi}\left(-\frac{\cos n\pi}{n} + \frac{\cos n\cdot 0}{n}\right) =$$

$$= \frac{2}{\pi}\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right) = \boxed{\frac{2}{\pi n}(1-(-1)^n)}$$

Opet ćemo razlikovati parne i neparne članove:

Za n paran broj je $\text{b}_{2n} = 0$

Za n neparan broj je $\text{b}_{2n-1} = \frac{2}{\pi(2n-1)}(1+1) = \frac{4}{\pi(2n-1)}$

Sada se vratimo u početnu formulu za razvoj i imamo:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)}\sin(2n-1)x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

$$\boxed{\text{sgn}\, x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}}$$

Primer 3.

Funkciju $f(x) = \{ \begin{matrix} \pi, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi \end{matrix}$ razviti u trigonometrijski red.

Rešenje:

Najpre uočimo da je zadati interval $[-\pi,\pi]$. Znači da ćemo koristiti formule:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Pazite na jednu stvar: pošto je funkcija zadata na ovaj način moramo raditi 2 integrala, gde ćemo kad su granice od $-\pi$ do 0 uzimati vrednost *f(x)* = π, a kad granice idu od 0 do π uzimamo *f(x)* = x

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi dx + \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{\pi}\pi\cdot x / {}_{-\pi}^{0} + \frac{1}{\pi}\frac{x^2}{2} / {}_0^{\pi} = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\not\pi}\int_{-\pi}^{0} \not\pi \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$$= \int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx$$

Ovde imamo integral sa parcijalnom integracijom, pa ćemo njegovu vrednost (bez granica naći “na stranu”)

$$\int x\cos nx dx = \begin{vmatrix} x = u & \cos nx dx = dv \\ dx = du & \frac{1}{n}\sin nx = v \end{vmatrix} = x\cdot\frac{1}{n}\sin nx - \int \frac{1}{n}\sin nx dx = \frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx dx =$$

$$= \frac{x\sin nx}{n} + \frac{1}{n}\frac{1}{n}\cos nx = \frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx$$

Sad se vratimo u a_n :

$$a_n = \int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx = \frac{1}{n}\sin nx \Big/_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right)\Big/_{0}^{\pi}$$

$$= [\underbrace{\frac{1}{n}\sin n\cdot 0 - \frac{1}{n}\sin n(-\pi)}_{ovo\ je\ sve\ 0}] + \frac{1}{\pi}[\left(\underbrace{\frac{\pi\sin n\pi}{n}}_{ovo\ je\ 0} + \frac{1}{n^2}\cos n\pi\right) - \left(\underbrace{\frac{0\sin n\cdot 0}{n}}_{ovo\ je\ 0} + \frac{1}{n^2}\cos n\cdot 0\right)]$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}((-1)^n - 1)$$

Dakle:

$$a_n = \frac{1}{\pi n^2}((-1)^n - 1) = \begin{cases} \frac{-2}{\pi(2k+1)^2}, & n = 2k+1, \quad k = 0,1,2,3... \\ 0, & n = 2k, \quad k = 0,1,2,3... \end{cases}$$

Još da nadjemo :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$

$$= \int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$

I ovde ćemo najpre odraditi parcijalnu integraciju:

$$\int x\sin nx dx = \begin{vmatrix} x = u & \sin nx dx = dv \\ dx = du & -\frac{1}{n}\cos nx = v \end{vmatrix} = -x\cdot\frac{1}{n}\cos nx + \int \frac{1}{n}\cos nx dx = -\frac{x\cos nx}{n} + \frac{1}{n}\int \cos nx dx =$$

$$= -\frac{x\cos nx}{n} + \frac{1}{n}\frac{1}{n}\sin nx = \boxed{-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx}$$

Sada imamo:

$$b_n = \int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx dx$$

$$= \left(-\frac{1}{n}\cos nx\right) / \begin{matrix} 0 \\ -\pi \end{matrix} + \frac{1}{\pi}\left(-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right) / \begin{matrix} \pi \\ 0 \end{matrix}$$

$$= \left\{\left(-\frac{1}{n}\cos n \cdot 0\right) - \left(-\frac{1}{n}\cos n(-\pi)\right)\right\} + \frac{1}{\pi}\left\{\left(-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin n\pi\right) - \left(-\frac{0\cdot\cos(n\cdot 0)}{n} + \frac{1}{n^2}\sin(n\cdot 0)\right)\right\}$$

$$= -\frac{1}{n} + \frac{1}{n}\cos n\pi + \frac{1}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right)$$

$$b_n = -\frac{1}{n} + \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n}$$

$$b_n = -\frac{1}{n}$$

Sada možemo zapisati i ceo razvoj:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\boxed{f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)\pi)}{(2k+1)^2} - \sum_{n=1}^{\infty}\frac{\sin nx}{n}}$$

Ovaj red konvergira ka funkciji S koja se, po Dirihleovoj teoremi poklapa sa funkcijom f na intervalu:

$[-\pi, 0) \cup (0, \pi]$ a kako f(x) ima prekid za x = 0 to je $S(0) = \frac{f(0-0)+f(0+0)}{2} = \frac{\pi+0}{2} = \frac{\pi}{2}$

grafik pogledajte na slici: