POVRŠINSKI INTEGRALI (zadaci- III deo)

STOKSOVA FORMULA

Ako su P, Q, R neprekidne diferencijabilne funkcije a L zatvorena , deo po deo glatka kriva koja je granica deo po deo dvostrane površi S, tada je:

$$\oint_L Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \iint_S [\cos\alpha(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) - \cos\beta(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + \cos\gamma(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})]dS$$

pri čemu su $\cos\alpha$, $\cos\beta$ i $\cos\gamma$ koordinate normale površi S koja je orijentisana na onu stranu u odnosu na koju se obilazak krive L vrši u suprotnom smeru od smera kretanja kazaljke na satu.

1. Primenom Stoksove formule izračunati $\int_C e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$ ako je C linija odredjena presekom površi $z = \sqrt{x^2+y^2}, x=0, x=2, y=0, y=1$.

Rešenje:

Ovde ćemo najpre pronaći parcijalne izvode koji nam trebaju!

Iz $\int_C e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$ **je**

$$P = e^x \to \frac{\partial P}{\partial y} = 0 \wedge \frac{\partial P}{\partial z} = 0$$

$$Q = z(x^2+y^2)^{\frac{3}{2}} \to \frac{\partial Q}{\partial x} = \frac{3}{2} z(x^2+y^2)^{\frac{1}{2}} \cdot (x^2+y^2)'_{po\ x} = \frac{3}{\cancel{2}} z(x^2+y^2)^{\frac{1}{2}} \cdot \cancel{2}x = xz(x^2+y^2)^{\frac{1}{2}}$$

$$\to \frac{\partial Q}{\partial z} = (x^2+y^2)^{\frac{3}{2}}$$

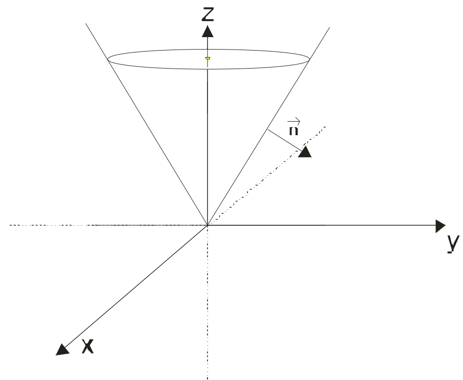
$$R = yz^3 \to \frac{\partial R}{\partial x} = 0 \wedge \frac{\partial R}{\partial y} = z^3$$

Dalje nam treba p i q:

$$z = \sqrt{x^2+y^2} \to p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \wedge q = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \text{ pa je onda:}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$$

Kako je u pitanju tup ugao(pogledajmo sliku)



Onda je : $\gamma > 90^0 \rightarrow \cos\gamma < 0 \rightarrow$ uzimamo plus ispred korena u imeniocu $\cos\gamma = \frac{-1}{+\sqrt{1+p^2+q^2}} = -\frac{1}{\sqrt{1+p^2+q^2}}$

$$\cos\gamma = \frac{-1}{+\sqrt{1+p^2+q^2}} = -\frac{1}{\sqrt{2}}$$

$$\cos\alpha = \frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{x}{\sqrt{x^2+y^2}}$$

$$\cos\beta = \frac{1}{\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}$$

Sad možemo da se vratimo u formulu:

$$\oint_L Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \iint_S [\cos\alpha(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) - \cos\beta(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + \cos\gamma(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})]dS$$

$$\iint_S [\cos\alpha(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) - \cos\beta(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + \cos\gamma(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})]dS =$$

$$\iint_S [\frac{1}{\sqrt{2}}\frac{x}{\sqrt{x^2+y^2}}(z^3 - (x^2+y^2)^{\frac{3}{2}}) - \frac{1}{\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}(0-0) - \frac{1}{\sqrt{2}}(3xz(x^2+y^2)^{\frac{1}{2}} - 0)]dS =$$

$$\iint_S [\frac{1}{\sqrt{2}}\frac{x}{\sqrt{x^2+y^2}}(z^3 - (x^2+y^2)^{\frac{3}{2}}) - \frac{1}{\sqrt{2}}xz(x^2+y^2)^{\frac{1}{2}}]dS =$$

$$\iint_D [\frac{1}{\sqrt{2}}\frac{x}{\sqrt{x^2+y^2}}(\underbrace{\boxed{\left(\sqrt{x^2+y^2}\right)^3 - (x^2+y^2)^{\frac{3}{2}})}}_{ovo\ je\ 0} - \frac{1}{\sqrt{2}}3x\sqrt{x^2+y^2}(x^2+y^2)^{\frac{1}{2}}]\cdot\sqrt{1+p^2+q^2}\,dxdy =$$

$$\iint_D [-\frac{1}{\cancel{\sqrt{2}}}3x(x^2+y^2)]\cdot\cancel{\sqrt{2}}\,dxdy = -3\iint_D x(x^2+y^2)dxdy$$

Granice oblasti integracije su nam date na početku zadatka, pa je $D:\begin{cases}0\le x\le 2\\ 0\le y\le 1\end{cases}$

$$-3\iint_D x(x^2+y^2)dxdy=-3\int_0^2 dx\int_0^1 (x^3+xy^2)dy=....=-14$$

2. **Primenom Stoksove formule izračunati** $\int_C (y-z)dx+(z-x)dy+(x-y)dz$ **ako je C luk elipse** $\begin{cases}x^2+y^2=a^2\\ \dfrac{x}{a}+\dfrac{z}{h}=1\end{cases}$

gde je $a>0$ **i h > 0 orijentisan u smeru suprotnom od smera kayaljke na časovniku, posmatrano sa pozitivnog smera ose Ox.**

Rešenje:

$$P=y-z\to\frac{\partial P}{\partial y}=1\wedge\frac{\partial P}{\partial z}=-1$$

$$Q=z-x\to\frac{\partial Q}{\partial x}=-1\wedge\frac{\partial Q}{\partial z}=1$$

$$R=x-y\to\frac{\partial R}{\partial x}=1\wedge\frac{\partial R}{\partial y}=-1$$

Iz $\frac{x}{a}+\frac{z}{h}=1$ je $z=h(1-\frac{x}{a})\to p=\frac{\partial z}{\partial x}=-\frac{h}{a}\wedge q=\frac{\partial z}{\partial y}=0$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(-\frac{h}{a}\right)^2}=\frac{\sqrt{a^2+h^2}}{a}$$

$$\cos\gamma=\frac{-1}{-\sqrt{1+p^2+q^2}}=\frac{a}{\sqrt{a^2+h^2}}$$

$$\cos\alpha=\frac{-\frac{h}{a}}{-\frac{\sqrt{a^2+h^2}}{a}}=\frac{h}{\sqrt{a^2+h^2}}$$

$$\cos\beta=0$$

Sad ovo ubacimo u formulu :

$$\int_C (y-z)dx+(z-x)dy+(x-y)dz=\iint_S[\cos\alpha(-1-1)-\cos\beta(1+1)+\cos\gamma(-1-1)]dS=$$

$$=-2\iint_S[\cos\alpha+\cos\beta+\cos\gamma]dS=-2\iint_S[\frac{h}{\sqrt{a^2+h^2}}+0+\frac{a}{\sqrt{a^2+h^2}}]dS=-2\iint_S\frac{a+h}{\sqrt{a^2+h^2}}dS$$

Kako su *a* i *h* konstante , ceo ovaj izraz ide ispred integrala:

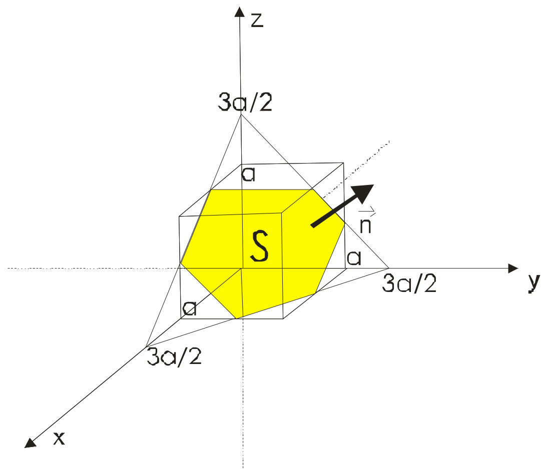
$$-2\iint_S\frac{a+h}{\sqrt{a^2+h^2}}dS=-2\frac{a+h}{\sqrt{a^2+h^2}}\iint_S dS=\frac{-2(a+h)}{\sqrt{a^2+h^2}}\iint_D\sqrt{1+p^2+q^2}dxdy=\frac{-2(a+h)}{\cancel{\sqrt{a^2+h^2}}}\iint_D\frac{\cancel{\sqrt{a^2+h^2}}}{a}dxdy=$$

$$=\frac{-2(a+h)}{a}\underbrace{\boxed{\iint_D dxdy}}_{\substack{\text{površina}\\ \text{oblastiD}\\ (\text{kruga})}}=\frac{-2(a+h)}{a}\cdot a^2\pi=\boxed{-2a(a+h)\pi}$$

3. Primenom Stoksove formule izračunati $\iint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$ ako je kriva C odredjena sa

$0 \le x \le a, \quad 0 \le y \le a, \quad 0 \le z \le a, \quad x + y + z = \frac{3a}{2}$

Rešenje:
Nacrtajmo najpre sliku….



Iz $\iint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz \rightarrow P = y^2 - z^2, Q = x^2 - y^2, R = x^2 - y^2$ pa je:

$$I = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & x^2 - y^2 & x^2 - y^2 \end{vmatrix} dS = \iint_S [\cos\alpha(-2y - 2z) - \cos\beta(2x + 2z) + \cos\gamma(-2x - 2y)]dS$$

$$x + y + z = \frac{3a}{2} \rightarrow z = \frac{3a}{2} - x - y \rightarrow p = \frac{\partial z}{\partial x} = -1 \wedge q = \frac{\partial z}{\partial y} = -1$$

$$\sqrt{1 + p^2 + q^2} = \sqrt{3}$$

Sa slike vidimo da: $\gamma < 90^0 \rightarrow \cos\gamma = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} = \cos\beta = \cos\gamma$

Vratimo se u zadatak:

$$I = \iint_S [\cos\alpha(-2y - 2z) - \cos\beta(2x + 2z) + \cos\gamma(-2x - 2y)]dS =$$

$$= \iint_S [\frac{-2}{\sqrt{3}}(y + z) - \frac{2}{\sqrt{3}}(x + z) - \frac{2}{\sqrt{3}}(x + y)]dS =$$

$$= \frac{-2}{\sqrt{3}} \iint_S [y + z + x + z + x + y]dS =$$

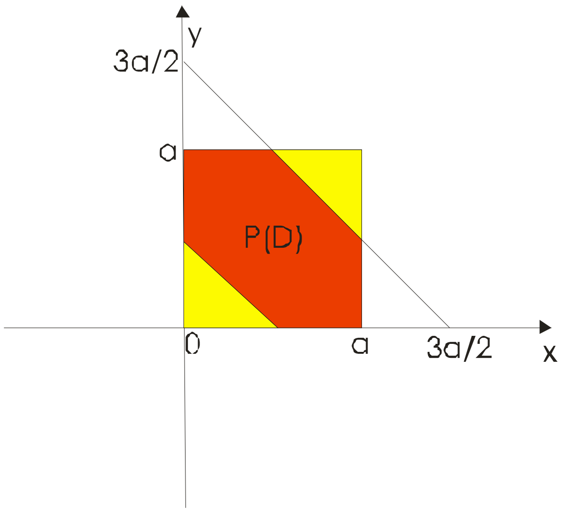
$$= \frac{-4}{\sqrt{3}} \iint_S [x + y + z]dS$$

$$= \frac{-4}{\sqrt{3}} \iint_D [x + y + \frac{3a}{2} - x - y]\sqrt{1 + p^2 + q^2}\,dxdy$$

$$= \frac{-4}{\cancel{\sqrt{3}}} \iint_D \frac{3a}{2} \cancel{\sqrt{3}}\,dxdy = -6a \iint_D dxdy = -6a \cdot P(D)$$

E, sad je pitanje: kolika je površina oblasti D?

Nacrtajmo sliku u ravni….



Površinu oblasti D ćemo dobiti kad od površine kvadrata oduzmemo površinu ova dva mala žuta trougla.

(a oni zajedno daju jednu četvrtinu površine kvadrata...)

Dakle: $P(D) = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

Pa će konačno rešenje biti:

$I = -6a \cdot P(D) = -6a \cdot \frac{3a^2}{4} = -\frac{9a^3}{2}$

FORMULA OSTROGRADSKOG

Ako je S deo po deo glatka površ , koja ograničava oblast V , a P, Q i R neprekidne funkcije zajedno sa svojim parcijalnim izvodima prvog reda u oblasti $V \cup S$, onda važi formula:

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz$$

to jest

$$\iint_S Pdydz + Qdxdz + Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz$$

gde su $\cos\alpha$, $\cos\beta$ i $\cos\gamma$ kosinusi pravca spoljašnje normale površi S.

4. Izračunati $\iint_S y^2zdxdy + xzdydz + x^2ydxdz$ **ako je S spoljna strana površi koju čine površi** $z = x^2 + y^2, \quad x^2 + y^2 = 1, \quad x = 0, \quad y = 0, \quad z = 0$ **u prvom oktantu.**

Rešenje:

Iz $\iint_S y^2zdxdy + xzdydz + x^2ydxdz$ pročitamo P,Q i R i nadjemo parcijalne izvode koji nam trebaju.

Ali pazite na redosled, ako vam je zgodnije poredjajte redom…. $\iint_S \boxed{xz}dydz + \boxed{x^2y}dxdz + \boxed{y^2z}dxdy$

$$P = xz \to \frac{\partial P}{\partial x} = z$$

$$Q = x^2y \to \frac{\partial Q}{\partial y} = x^2$$

$$R = y^2z \to \frac{\partial R}{\partial z} = y^2$$

Sad ovo ubacimo u formulu:

$$\iint_S Pdydz + Qdxdz + Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz = \iiint_V (z + x^2 + y^2)dxdydz$$

Sad trebamo rešiti ovaj trostruki integral:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \to |J| = r \\ z = z \end{cases}$$

Iz $x^2 + y^2 = 1 \to r^2 = 1 \to r = 1 \to 0 \le r \le 1$.

Kako se radi o prvom oktantu, onda je $0 \le \varphi \le \frac{\pi}{2}$

Iz $z = x^2 + y^2 \to z = r^2 \to 0 \le z \le r^2$

$$\iiint_V (z + x^2 + y^2)dxdydz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr \int_0^{r^2} (z + r^2) \cdot rdz = \underbrace{\boxed{\int_0^{\frac{\pi}{2}} d\varphi}}_{\text{Ovo je odmah } \frac{\pi}{2}} \int_0^1 dr \int_0^{r^2} (zr + r^3)dz = ... = \frac{\pi}{8}$$

5. Izračunati $\iint_S xdydz + ydxdz + zdxdy$ **ako je S spoljna strana površi koju ograničava telo :**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \wedge \frac{x^2}{a^2}+\frac{z^2}{c^2}\le 1$$

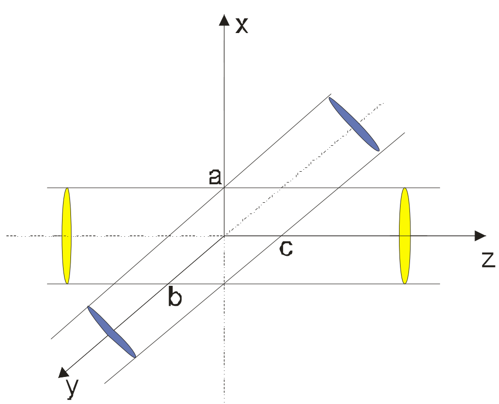
Rešenje:

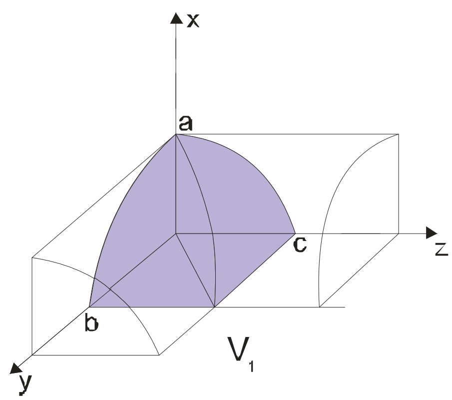
Iz $\iint_S xdydz + ydxdz + zdxdy \to P = x, Q = y, R = z$ pa je : $\frac{\partial P}{\partial x}=1;\quad \frac{\partial Q}{\partial y}=1;\quad \frac{\partial R}{\partial z}=1$

Ubacimo ovo u formulu i dobijamo:

$$\iint_S xdydz + ydxdz + zdxdy = \iiint_V (1+1+1)dxdydz = 3\iiint_V dxdydz$$

Nacrtajmo sada sliku (radi se o cilindrima).





Mi ćemo posmatrati situaciju u prvom oktantu, pa ćemo dobijeno rešenje pomnožiti sa 8.

$$3\iiint_V dxdydz = 3\cdot 8\cdot \iiint_{V_1} dxdydz = 24\iiint_{V_1} dxdydz$$

Da odredimo granice:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \wedge y=0 \to \frac{x^2}{a^2}\le 1 \to x^2 = a^2 \to \boxed{0\le x\le a}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \to y^2 = b^2\left(1-\frac{x^2}{a^2}\right) \to y = b\sqrt{1-\frac{x^2}{a^2}} \to \boxed{0\le y\le b\sqrt{1-\frac{x^2}{a^2}}}$$

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1 \to \frac{z^2}{c^2}=1-\frac{x^2}{a^2} \to \boxed{0\le z\le c\sqrt{1-\frac{x^2}{a^2}}}$$

Vraćamo se da rešimo integral:

$$24\cdot\iiint_{V_1} dxdydz = 24\int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_0^{c\sqrt{1-\frac{x^2}{a^2}}} dz = ... = 16abc$$