POVRŠINSKI INTEGRALI (zadaci- II deo)

Da se podsetimo!

Ako je S glatka dvostrana površ na kojoj je izabrana jedna od dveju strana , određena smerom normale

$\vec{n}(\cos\alpha, \cos\beta, \cos\gamma)$ i z = z(x,y) tada je :

$$\cos\alpha = \frac{p}{\pm\sqrt{1+p^2+q^2}}$$

$$\cos\beta = \frac{q}{\pm\sqrt{1+p^2+q^2}} \qquad \text{gde je: } \quad p = \frac{\partial z}{\partial x} \quad \text{i} \quad q = \frac{\partial z}{\partial y}$$

$$\cos\gamma = \frac{-1}{\pm\sqrt{1+p^2+q^2}}$$

VAŽNO

(**Da li ćemo uzeti + ili – zavisi od ugla koji normala gradi sa pozitivnim delom z-ose:**

Ako je taj ugao oštar ,onda mora biti cos γ **>0 pa uzimamo minus ispred korena,** $\cos\gamma = \frac{-1}{-\sqrt{1+p^2+q^2}}$

Ako je taj ugao tup, onda je cos γ **<0 , pa uzimamo + ispred korena** $\cos\gamma = \frac{-1}{+\sqrt{1+p^2+q^2}}$ **)**

a P=P(x,y,z) Q=Q(x,y,z) i R=R(x,y,z) tri funkcije, definisane i neprekidne na površi S, onda je

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

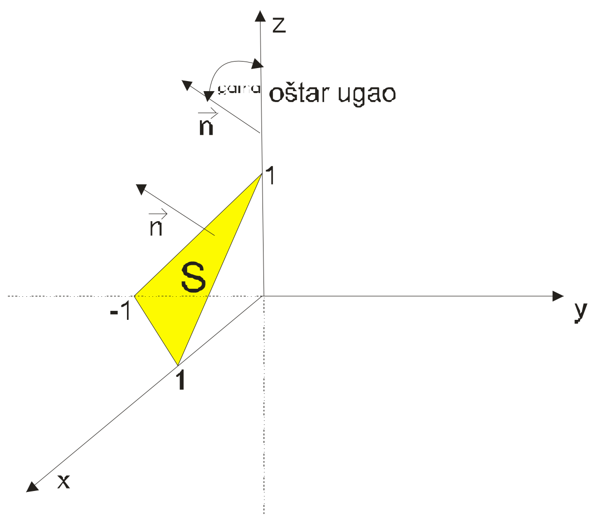
Površinski integral druge vrste **zavisi** od orijentacije krive.

Prelaskom na drugu stranu površi menja se znak !.

1. Izračunati površinski integral $I=\iint_S zdxdy+xdxdz+ydydz$ **ako je S gornji deo ravni** $x-y+z=1$ **isečen koordinatnim ravnima.**

Rešenje:

Nacrtajmo najpre sliku:



Iz date ravni izrazimo z pa nadjemo p i q.

$$x-y+z=1\to z=1-x+y\to p=\frac{\partial z}{\partial x}=-1\wedge q=\frac{\partial z}{\partial y}=1$$

Sa slike vidimo da je ugao izmedju normale na ravan i z ose (pozitivan smer gledamo) oštar, pa nam to govori da ćemo ispred korena u imeniocu uzimati minus! (To jest, pravimo da ceo $\cos\gamma=\dfrac{-1}{\pm\sqrt{1+p^2+q^2}}$ bude pozitivan)

$$\cos\gamma=\frac{-1}{-\sqrt{1+p^2+q^2}}=\frac{1}{\sqrt{1+(-1)^2+1^2}}=\frac{1}{\sqrt{3}}$$

Kad smo zaključili da je minus ispred korena, tako radimo i za ostala dva:

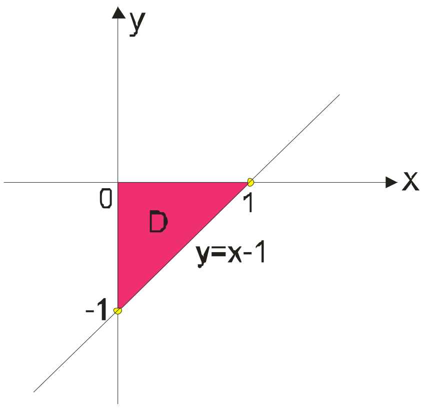
$$\cos\alpha=\frac{p}{\pm\sqrt{1+p^2+q^2}}=\frac{-1}{-\sqrt{1+(-1)^2+1^2}}=\frac{1}{\sqrt{3}}$$
$$\cos\beta=\frac{q}{\pm\sqrt{1+p^2+q^2}}=\frac{1}{-\sqrt{1+(-1)^2+1^2}}=-\frac{1}{\sqrt{3}}$$

Iz datog integrala $I=\iint_S zdxdy+xdxdz+ydydz$ **uporedjujući ga sa** $\iint_S Pdydz+Qdzdx+Rdxdy$ pročitamo da je P = y , Q= x i R= z.

Sad možemo upotrebiti formulu: $\iint\limits_S Pdydz + Qdzdx + Rdxdy = \iint\limits_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$

$$I = \iint\limits_S zdxdy + xdxdz + ydydz = \iint\limits_S (y\cdot\frac{1}{\sqrt{3}} + x\cdot\left(-\frac{1}{\sqrt{3}}\right) + z\cdot\frac{1}{\sqrt{3}})dS = \frac{1}{\sqrt{3}}\iint\limits_S (y - x + z)dS$$

Vreme je da problem spustimo u ravan z = 0, nacrtamo sliku i odredimo granice.



$$z = 0 \wedge x - y + z = 1 \rightarrow x - y = 1 \rightarrow y = x - 1$$

$D:\begin{cases} 0 \le x \le 1 \\ x - 1 \le y \le 0 \end{cases}$ I već smo videli da je: $\sqrt{1 + p^2 + q^2} = \sqrt{1 + (-1)^2 + 1^2} = \sqrt{3}$

$$I = \frac{1}{\sqrt{3}}\iint\limits_S (y - x + z)dS = \frac{1}{\sqrt{3}}\iint\limits_S (y - x + 1 - x - y)\sqrt{1 + p^2 + q^2}dxdy = \frac{1}{\sqrt{3}}\iint\limits_S (2y - 2x + 1)\sqrt{3}\,dxdy =$$

$$= \int_0^1 dx \int_{x-1}^0 (2y - 2x + 1)dy = ... = -\frac{1}{6}$$

<u>II način</u>

Neki profesori <u>ne vole</u> da rade ovako, već početni integral podele na tri integrala:

$$I = \iint\limits_S zdxdy + xdxdz + ydydz = \underbrace{\boxed{\iint\limits_S zdxdy}}_{I_1} + \underbrace{\boxed{\iint\limits_S xdxdz}}_{I_2} + \underbrace{\boxed{\iint\limits_S ydydz}}_{I_3}$$

E sad svaki integral rešavamo posebno pa ćemo sabrati rešenja!

Za prvi integral koji radimo po dxdy izrazimo z i zamenimo, a granice odredjujemo u z = 0

$$I_1 = \iint\limits_S zdxdy = \iint\limits_S (1 + y - x)dxdy = \int_0^1 dx \int_{x-1}^0 (1 + y - x)dy = ... = \frac{1}{6}$$

Za drugi integral koji radimo po dxdz izrazimo *y* i zamenimo(ovde to ne mora, jer imamo samo x u integralu)

a granice odredjujemo u y = 0

$$I_2 = \iint\limits_S xdxdz = -\int_0^1 xdx \int_0^{1-x} dz = ... = -\frac{1}{6}$$

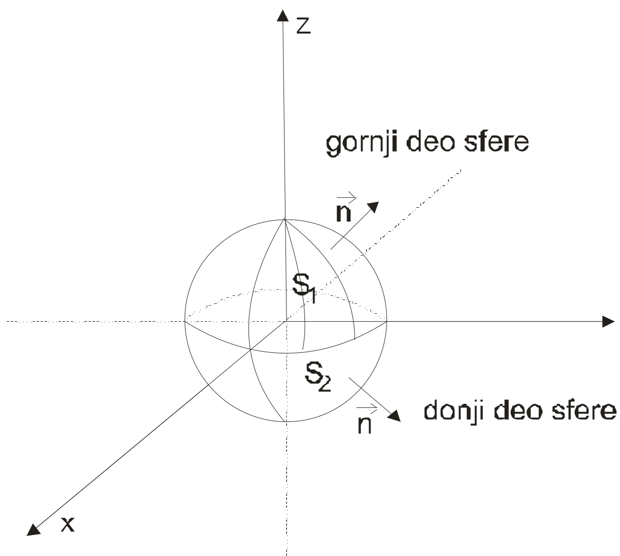
Za treći slično, pa imamo $I_3 = \iint_S ydydz = \int_{-1}^{0} ydy \int_{0}^{y+1} dz = ... = -\frac{1}{6}$

Kad ih saberemo, dobijamo rezultat $I = \iint_S zdxdy + xdxdz + ydydz = \underbrace{\boxed{\iint_S zdxdy}}_{I_1} + \underbrace{\boxed{\iint_S xdxdz}}_{I_2} + \underbrace{\boxed{\iint_S ydydz}}_{I_3} = \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{6}$

2. Izračunati površinski integral $I = \iint_S xyzdxdy$ **ako je S spoljna strana površi** $x^2 + y^2 + z^2 = 1$ **,** $x \geq 0, y \geq 0$

Rešenje:

Nacrtajmo najpre sliku da vidimo kakav ugao gradi normala....



Šta primećujemo?

Za gornji deo sfere $z_1 = \sqrt{1 - x^2 - y^2}$ normala pravi oštar ugao sa pozitivnim smerom z ose a za donji deo sfere,

$z_2 = -\sqrt{1 - x^2 - y^2}$ normala pravi tup ugao sa pozitivnim smerom z ose.

Zaključujemo da moramo posebno raditi za gornji deo, posebno za donji, pa ćemo sabrati rešenja.

<u>ZA S_1</u>

Kako je $\gamma < 90^0 \rightarrow \cos\gamma > 0 \rightarrow$ uzimamo minus ispred korena u imeniocu $\cos\gamma = \frac{-1}{-\sqrt{1 + p^2 + q^2}} = \frac{1}{\sqrt{1 + p^2 + q^2}}$

Da nadjemo sada ove parcijalne izvode i koren na stranu …

$$p=\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{1-x^2-y^2}}\wedge q=\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2}=\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}=$$

$$=\sqrt{\frac{1\cancel{-x^2}\cancel{-y^2}\cancel{+x^2}\cancel{+y^2}}{1-x^2-y^2}}=\frac{1}{\sqrt{1-x^2-y^2}}$$

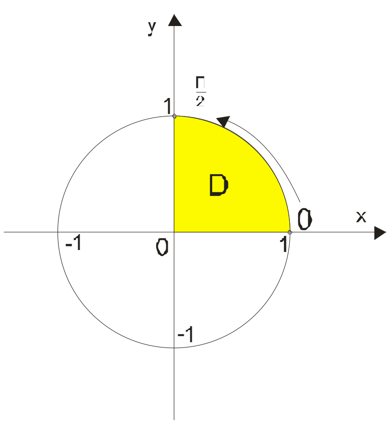
Pa je $\cos\gamma=\dfrac{1}{\sqrt{1+p^2+q^2}}=\sqrt{1-x^2-y^2}$

$$I_1=\iint_{S_1}xyzdxdy=\iint_{S_1}xyz\cos\gamma dxdy=\iint_{S_1}xy\sqrt{1-x^2-y^2}\cdot\sqrt{1-x^2-y^2}dxdy=\iint_{S_1}xy\left(1-x^2-y^2\right)dxdy$$

Spuštamo se u ravan z=0 , imamo:

$$z=0\to x^2+y^2=1, x\geq 0\wedge y\geq 0$$

Moramo opet sliku:



Nastavljamo sa zadatkom:

$$I_1=\iint_{S_1}xy\left(1-x^2-y^2\right)dxdy=\iint_{D}xy\left(1-x^2-y^2\right)\sqrt{1+p^2+q^2}dxdy=\iint_{D}xy\left(1-x^2-y^2\right)\frac{1}{\sqrt{1-x^2-y^2}}dxdy=$$

Sad je najbolje da uzmemo:

$$\left|\begin{array}{l} x=r\cos\varphi\wedge y=r\sin\varphi\to|J|=r \\ x^2+y^2=1\to r=1\to D:\begin{cases}0\leq r\leq 1\\ 0\leq\varphi\leq\dfrac{\pi}{2}\end{cases}\end{array}\right|$$

$$I_1=\iint_D xy\left(1-x^2-y^2\right)\frac{1}{\sqrt{1-x^2-y^2}}dxdy=\int_0^{\frac{\pi}{2}}d\varphi\int_0^1 r^2\sin\varphi\cos\varphi\cdot\sqrt{1-r^2}\cdot rdr=$$

$$=\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^1 r^3\cdot\sqrt{1-r^2}\,dr=$$

Integral sa uglom spakujemo $\sin\varphi\cos\varphi=\frac{1}{2}\sin 2\varphi$ a u ovom sa r uzmemo smenu $1-r^2=t^2$ i dobijamo: $\boxed{I_1=\frac{1}{15}}$

Sad radimo za donji deo sfere.

ZA S_2 (ovde je $z_2=-\sqrt{1-x^2-y^2}$)

Kako je $\gamma>90^0\to\cos\gamma<0\to$ uzimamo plus ispred korena u imeniocu $\cos\gamma=\frac{-1}{+\sqrt{1+p^2+q^2}}=-\frac{1}{\sqrt{1+p^2+q^2}}$

$$p=\frac{\partial z}{\partial x}=\frac{x}{\sqrt{1-x^2-y^2}}\wedge q=\frac{\partial z}{\partial y}=\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{y}{\sqrt{1-x^2-y^2}}\right)^2}=\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}=$$

$$=\sqrt{\frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2}}=\frac{1}{\sqrt{1-x^2-y^2}}$$

Pa je $\cos\gamma=-\frac{1}{\sqrt{1+p^2+q^2}}=-\sqrt{1-x^2-y^2}$

$$I_2=\iint_{S_2}xyzdxdy=\iint_{S_2}xyz\cos\gamma dxdy=\iint_{S_2}xy\left(-\sqrt{1-x^2-y^2}\right)\cdot\left(-\sqrt{1-x^2-y^2}\right)dxdy=\iint_{S_2}xy\left(1-x^2-y^2\right)dxdy=I_1$$

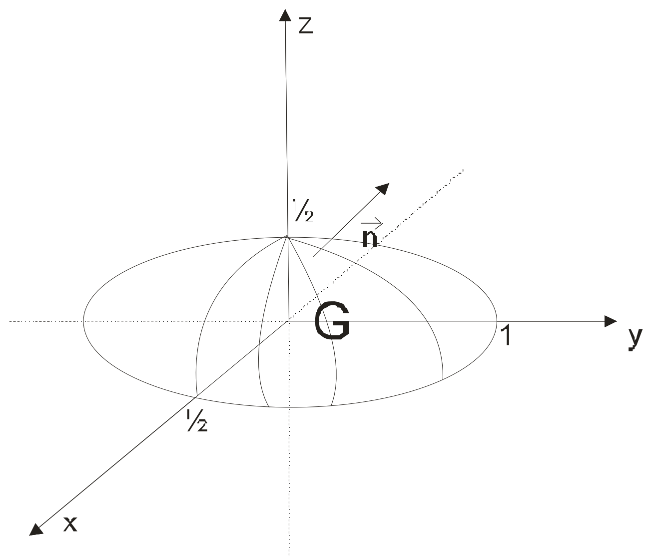
Dakle, vrednost ovog dela je takodje $\boxed{I_2=I_1=\frac{1}{15}}$ pa je konačno: $\boxed{I=I_2+I_1=\frac{2}{15}}$

3. Izračunati integral $\iint\limits_G 2dxdy + ydzdx - x^2zdydz$ **gde je G spoljna strana elipsoida** $4x^2 + y^2 + 4z^2 = 1$ **koji pripada prvom oktantu.**

Rešenje:

Kao i uvek prvo da nacrtamo sliku i odredimo smer normale:

$$4x^2 + y^2 + 4z^2 = 1 \rightarrow \frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \rightarrow a = \frac{1}{2}, b = 1, c = \frac{1}{2}$$



Ovaj zadatak ćemo pokušati da vam objasnimo na onaj drugi način, to jest zadati integral ćemo raditi kao tri zasebna integrala pa ćemo sabrati rešenja.

$$\iint\limits_G 2dxdy + ydzdx - x^2zdydz = \iint\limits_G 2dxdy + \iint\limits_G ydzdx + \iint\limits_G -x^2zdydz = I_1 + I_2 + I_3$$

$$\boxed{I_1 = \iint\limits_G 2dxdy}$$

Ovde imamo $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge z = 0 \rightarrow \frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} = 1$

$$I_1 = \iint\limits_{G^+} 2dxdy = 2\iint\limits_D dxdy = 2 \cdot m(D) = 2 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot 1 \cdot \pi = \frac{\pi}{4}$$

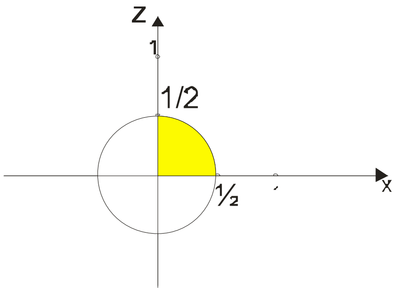
Samo da vas podsetimo da se površina elipse računa po formuli $P_{elipse} = ab\pi$ a ovde se radi o $\frac{1}{4}$ te površine.

$$\boxed{I_2 = \iint\limits_G ydzdx}$$

$$4x^2 + y^2 + 4z^2 = 1 \rightarrow y^2 = 1 - 4x^2 - 4z^2 \rightarrow y = \sqrt{1 - 4x^2 - 4z^2}$$

$$I_2 = \iint_{G^+} ydzdx = \iint_D \sqrt{1-4x^2-4z^2}\,dzdx$$

Kako je ovde $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge y = 0 \to \frac{x^2}{\frac{1}{4}} + \frac{z^2}{\frac{1}{4}} = 1 \to x^2 + z^2 = \frac{1}{4}$ Odlast D će biti (pogledajmo sliku):



$$\left.\begin{aligned} x &= r\cos\varphi \\ z &= r\sin\varphi \end{aligned}\right\} \to |J| = r \wedge r^2 = \frac{1}{4} \to r = \frac{1}{2} \to 0 \le r \le \frac{1}{2} \wedge 0 \le \varphi \le \frac{\pi}{2}$$

$$I_2 = \iint_{G^+} ydzdx = \iint_D \sqrt{1-4x^2-4z^2}\,dzdx = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{1}{2}} \sqrt{1-4r^2} \cdot rdr = ... = \frac{\pi}{24}$$

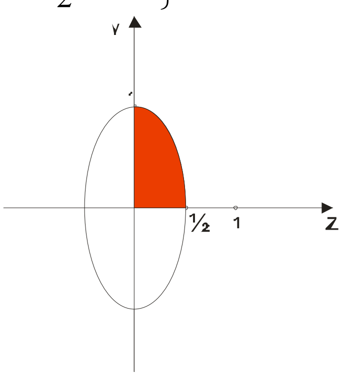
$$\boxed{I_3 = \iint_G -x^2 zdydz}$$

$$4x^2 + y^2 + 4z^2 = 1 \to x^2 = \frac{1}{4}\left(1-y^2-4z^2\right) \to x = \frac{1}{2}\sqrt{1-y^2-4z^2}$$

$$I_3 = \iint_{G^+} -x^2 zdydz = \iint_D -\frac{1}{4}\left(1-y^2-4z^2\right)zdydz = -\frac{1}{4}\iint_D \left(1-y^2-4z^2\right)zdydz$$

Kako je $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge x = 0 \to \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \to y^2 + 4z^2 = 1$ to ćemo uzeti:

$$\left.\begin{aligned} y &= r\cos\varphi \\ z &= \frac{1}{2}r\sin\varphi \end{aligned}\right\} \to |J| = \frac{1}{2}r \wedge r^2 = 1 \to r = 1 \to 0 \le r \le 1 \wedge 0 \le \varphi \le \frac{\pi}{2}$$



$$I_3 = -\frac{1}{4}\iint_D \left(1-y^2-4z^2\right)zdydz = -\frac{1}{4}\int_0^{\frac{\pi}{2}} d\varphi \int_0^1 \frac{1}{2} r\sin\varphi\left(1-r^2\right)\frac{1}{2}rdr = ... = -\frac{1}{120}$$

Sad saberemo sva tri rešenja:

$$I = I_1 + I_2 + I_3 = \frac{\pi}{4} + \frac{\pi}{24} - \frac{1}{120} = \boxed{\frac{7\pi}{24} - \frac{1}{120}}$$

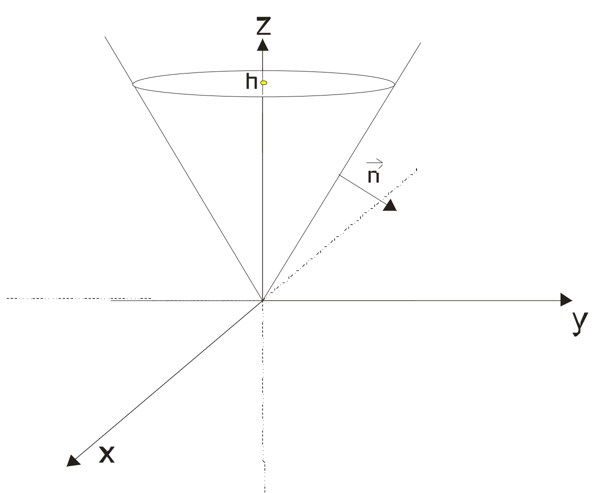
Vama za trening ostavljamo da zadatak rešite na onaj “prvi” način!

Uostalom, najbolje je da radite onako kako zahteva vaš profesor!

4. Izračunati integral $I = \iint_S (y-z)dydz + (z-x)dxdz + (x-y)dxdy$ ako je S spoljna strana površi $x^2 + y^2 = z^2 \wedge 0 \le z \le h$

Rešenje:

Dakle, ovde se radi o konusu, pogledajmo sliku....



Iz $x^2 + y^2 = z^2 \wedge 0 \le z \le h \to z = \sqrt{x^2 + y^2}$ pa je onda:

$$p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \wedge q = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{1+x^2+y^2}}\right)^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{\frac{2\,\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}} = \sqrt{2}$$

Sa slike vidimo da normala na spoljnu stranu konusa gradi sa pozitivnim smerom z ose tup ugao pa je: $\gamma > 90^0 \to \cos\gamma < 0 \to$ uzimamo **plus** ispred korena u imeniocu $\cos\gamma = \frac{-1}{+\sqrt{1+p^2+q^2}} = -\frac{1}{\sqrt{1+p^2+q^2}} = -\frac{1}{\sqrt{2}}$

a onda je

$$\cos\alpha = \frac{p}{+\sqrt{1+p^2+q^2}} = \frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{2}} =$$

$$\cos\beta = \frac{q}{+\sqrt{1+p^2+q^2}} = \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{2}} =$$

Sad upotrebljavamo formulu:

$$I = \iint_S (y-z)dydz + (z-x)dxdz + (x-y)dxdy = \iint_S [(y-z)\cos\alpha + (z-x)\cos\beta + (x-y)\cos\gamma]dS =$$

$$= \iint_S [(y-\sqrt{x^2+y^2})\frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{2}} + (\sqrt{x^2+y^2}-x)\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{2}} + (x-y)(-\frac{1}{\sqrt{2}})]dS =$$

$$= \iint_S [\frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{2}} - \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{2}} + (y-x)\frac{1}{\sqrt{2}}]dS = \iint_S 2(y-x)\frac{1}{\sqrt{2}}dS = \iint_S \sqrt{2}(y-x)dS = \sqrt{2}\iint_D (y-x)\sqrt{1+p^2+q^2}dxdy$$

Sad spustimo problem u ravan z = 0 i odredimo granice:

$$D: x^2+y^2 \le h^2$$

$$\left.\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}\right\} \to |J| = r \to D: \begin{cases} 0 \le r \le h \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I = \iint_S \sqrt{2}(y-x)dS = \sqrt{2}\int_0^{2\pi} d\varphi \int_0^h (r\sin\varphi - r\cos\varphi)\sqrt{2}rdr = 2\int_0^{2\pi} (\sin\varphi - \cos\varphi)d\varphi \int_0^h r^2 dr = ... = 0$$