PIRAMIDA I ZARUBLJENA PIRAMIDA

Slično kao i kod prizme i ovde ćemo najpre objasniti oznake ...

- sa ***a*** obeležavamo dužinu osnovne ivice
- sa **H** obeležavamo dužinu visine piramide
- sa **h** obeležavamo dužinu visine bočne strane (**apotema**)
- sa **s** obeležavamo dužinu bočne ivice
- sa **B** obeležavamo površinu osnove (baze)
- sa **M** obeležavamo površinu omotača
- omotač se sastoji od **bočnih strana**(najčešće jednakokraki trouglovi) , naravno trostrana piramida u omotaču ima 3 takve strane, četvorostrana - 4 itd.

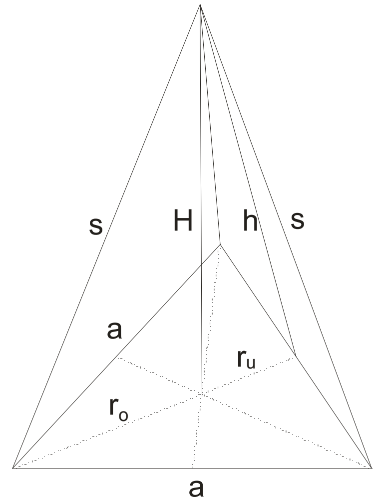
- ako u tekstu zadatka kaže **jednakoivična** piramida, to nam govori da su osnovna ivica i bočna ivica jednake , to jest : **a = s**
- ako u tekstu zadatka ima reč **prava** – to znači da je visina piramide normalna na ravan osnove ili ti , jednostavnije rečeno , piramida nije kriva
- ako u tekstu zadatka ima reč **pravilna** , to nam govori da je u osnovi (bazi) pravilan mnogougao: jednakostraničan trougao, kvadrat, itd.

Dve najvažnije formule za izračunavanje površine i zapremine su:

$$P = B + M \quad \text{za površinu i}$$

$$V = \frac{1}{3}\,\mathrm{B}\cdot\mathrm{H} \quad \text{za zapreminu}$$

PRAVA PRAVILNA TROSTRANA PIRAMIDA



Kako je u bazi jednakostraničan trougao, to će površina baze biti: $B = \frac{a^2\sqrt{3}}{4}$

U omotaču se nalaze tri jednakokraka trougla (površina jednog od njih je $P_{bočne\ strane} = \frac{a \cdot h}{2}$) , a kako ih ima 3 u omotaču, to je: $M = 3\frac{a \cdot h}{2}$

$$P = B + M$$

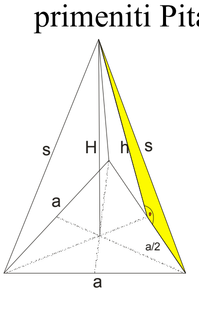
$$P = \frac{a^2\sqrt{3}}{4} + 3\frac{a \cdot h}{2}$$

$$V = \frac{1}{3}B \cdot H$$

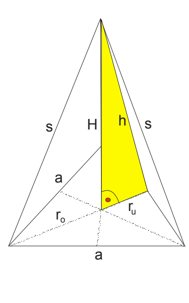
$$V = \frac{1}{3}\frac{a^2\sqrt{3}}{4} \cdot H$$

$$V = \frac{a^2\sqrt{3}}{12} \cdot H$$

Dalje nam trebaju primene Pitagorine teoreme . Kod svake piramide postoje po tri trougla na kojima možemo primeniti Pitagorinu teoremu:

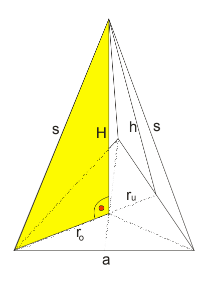


$$s^2 = h^2 + \left(\frac{a}{2}\right)^2$$



$h^2 = H^2 + r_u^2$ to jest

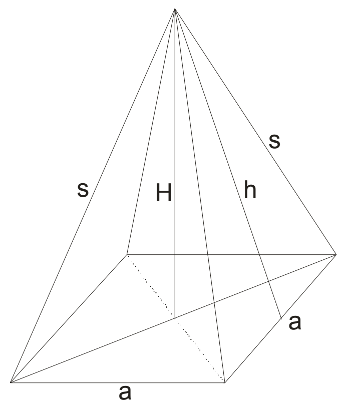
$$h^2 = H^2 + \left(\frac{a\sqrt{3}}{6}\right)^2$$



$s^2 = H^2 + r_o^2$ to jest

$$s^2 = H^2 + \left(\frac{a\sqrt{3}}{3}\right)^2$$

PRAVA PRAVILNA ČETVOROSTRANA PIRAMIDA



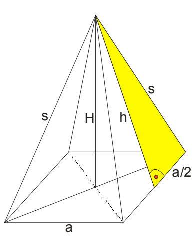
U bazi je kvadrat, pa je površina baze $B = a^2$

U omotaču se nalaze četiri jednakokraka trougla (površina jednog od njih je $P_{bočne\ strane} = \frac{a \cdot h}{2}$), pa je površina omotača $M = 4\frac{a \cdot h}{2}$ odnosno $M = 2ah$

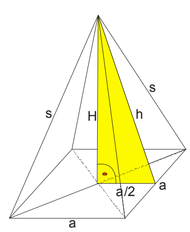
$$P = B + M \qquad V = \frac{1}{3} B \cdot H$$

$$P = a^2 + 2ah \qquad V = \frac{1}{3} a^2 \cdot H$$

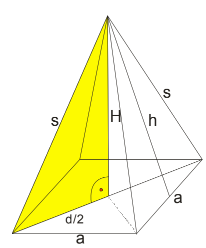
Primena Pitagorine teoreme:



$$s^2 = h^2 + \left(\frac{a}{2}\right)^2$$



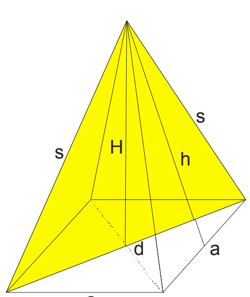
$$h^2 = H^2 + \left(\frac{a}{2}\right)^2$$



$s^2 = H^2 + \left(\frac{d}{2}\right)^2$ odnosno

$s^2 = H^2 + \left(\frac{a\sqrt{2}}{2}\right)^2$ to jest

$s^2 = H^2 + \frac{a^2}{2}$

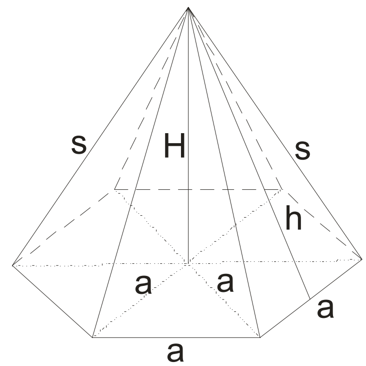


dijagonalni presek

$P_{DP} = \frac{d \cdot H}{2}$ odnosno

$P_{DP} = \frac{a \cdot H\sqrt{2}}{2}$

PRAVA PRAVILNA ŠESTOSTRANA PIRAMIDA



U bazi je šestougao, pa je površina baze $B = 6\frac{a^2\sqrt{3}}{4} = 3\frac{a^2\sqrt{3}}{2}$

U omotaču se nalaze šest jednakokraka trougla (površina jednog od njih je $P_{bočne\ strane} = \frac{a \cdot h}{2}$), pa je površina omotača jednaka $M = 6\frac{ah}{2} = 3ah$

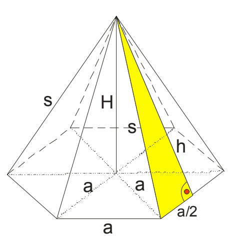
$$P = B + M$$

$$P = 3\frac{a^2\sqrt{3}}{2} + 3ah$$

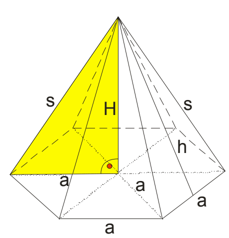
$$V = \frac{1}{3}BH$$

$$V = \frac{1}{3} \cdot 3\frac{a^2\sqrt{3}}{2}H$$

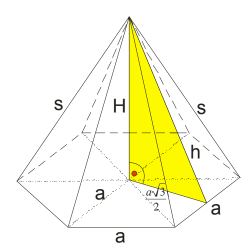
$$V = \frac{a^2\sqrt{3}}{2}H$$



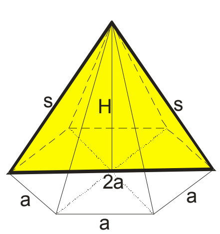
$$s^2 = h^2 + \left(\frac{a}{2}\right)^2$$



$$s^2 = H^2 + a^2$$



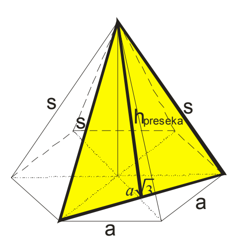
$$h^2 = H^2 + \left(\frac{a\sqrt{3}}{2}\right)^2$$



veći dijagonalni presek

P ovog dijagonalnog preseka je :

$P_{vdp} = \frac{2a \cdot H}{2}$ to jest $P_{vdp} = a \cdot H$



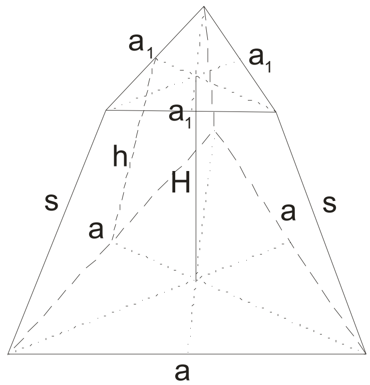
manji dijagonalni presek

P ovog dijagonalnog preseka je :

$$P_{mdp} = \frac{a\sqrt{3} \cdot h_{preseka}}{2}$$

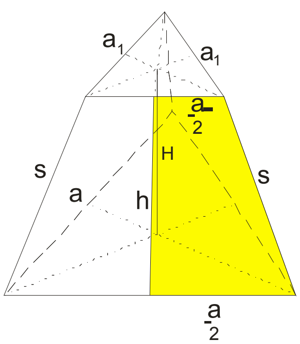
Može da se desi da u bazi nije pravilan mnogougao. Onda morate da sklapate bazu i omotač preko formulica za P trougla, romba, pravougaonika,......
Pogledajte formulice iz oblasti mnogougao , trouglovi i četvorouglovi....

PRAVA PRAVILNA TROSTRANA ZARUBLJENA PIRAMIDA

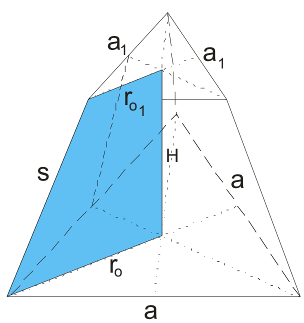


$P = B + B_1 + M$ $\qquad B = \dfrac{a^2\sqrt{3}}{4}$ $\qquad B_1 = \dfrac{a_1^{\,2}\sqrt{3}}{4}$ $\qquad M = 3\dfrac{a+a_1}{2}h$

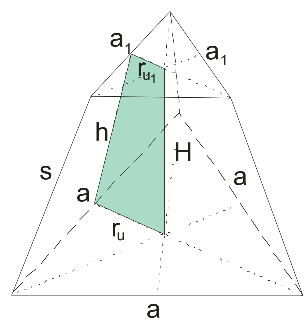
$V = \dfrac{H}{3}(B + B_1 + \sqrt{BB_1})$ ili $V = \dfrac{\sqrt{3}H}{12}(a^2 + a_1^2 + aa_1)$



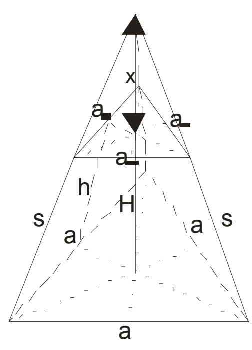
$\left(\dfrac{a-a_1}{2}\right)^2 + h^2 = s^2$



$\left(\dfrac{(a-a_1)\sqrt{3}}{3}\right)^2 + H^2 = s^2$

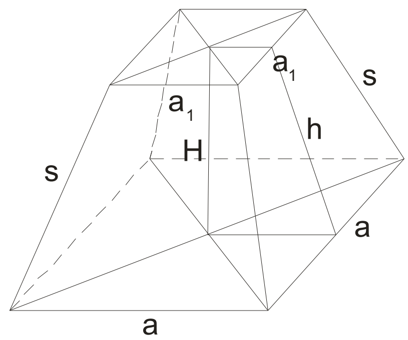


$\left(\dfrac{(a-a_1)\sqrt{3}}{6}\right)^2 + H^2 = h^2$



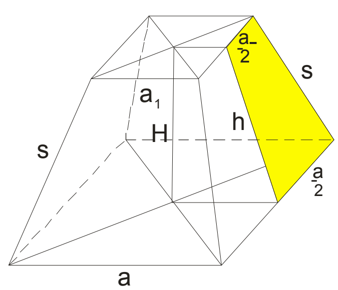
Visina dopunske piramide je: $x = \dfrac{\sqrt{B_1}H}{\sqrt{B} - \sqrt{B_1}}$

PRAVA PRAVILNA ČETVOROSTRANA ZARUBLJENA PIRAMIDA

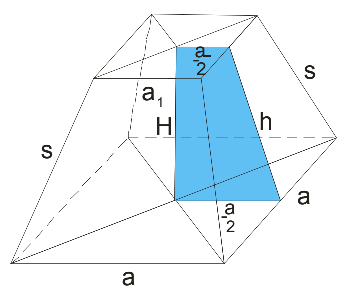


P = $B+B_1+M$ $\quad B=a^2 \quad B_1=a_1^2 \quad M=4\frac{a+a_1}{2}h=2(a+a_1)h$

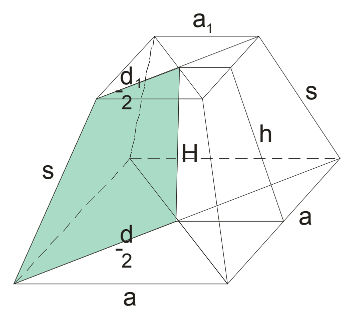
$V=\frac{H}{3}(B+B_1+\sqrt{BB_1}) \qquad V=\frac{H}{3}(a^2+a_1^2+aa_1)$



$(\frac{a-a_1}{2})^2+h^2=s^2$

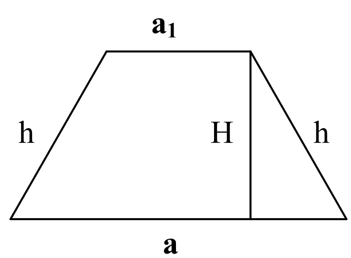


$(\frac{a-a_1}{2})^2+H^2=h^2$

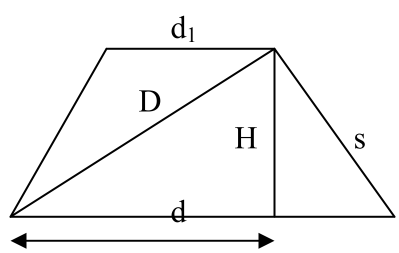


$(\frac{d-d_1}{2})^2+H^2=s^2$

osni presek:



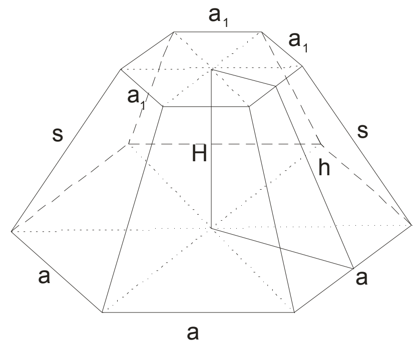
dijagonalni presek:



$\frac{d+d_1}{2}$

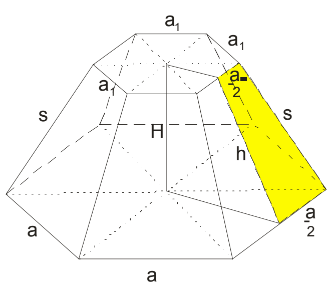
Ako sa x obeležimo visinu dopunske piramide , onda je $x=\frac{\sqrt{B_1}H}{\sqrt{B}-\sqrt{B_1}}=\frac{a_1H}{a-a_1}$

PRAVA PRAVILNA ŠESTOSTRANA ZARUBLJENA PIRAMIDA

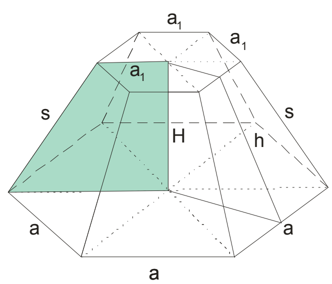


P = B+B_1+ M B=$\frac{6a^2\sqrt{3}}{4}$ B_1=$\frac{6a_1^{\ 2}\sqrt{3}}{4}$ M = $6\frac{a+a_1}{2}h$ =$3(a+a_1)h$

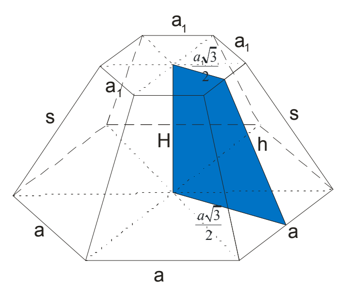
V= $\frac{H}{3}$(B+B_1+$\sqrt{BB_1}$) ili V= $\frac{H\sqrt{3}}{2}$(a^2+a_1^2+ aa_1)



$(\frac{a-a_1}{2})^2+h^2=s^2$



$(a-a_1)^2+H^2=s^2$



$(\frac{(a-a_1)\sqrt{3}}{2})^2+H^2=h^2$

Visina dopunske piramide je i ovde: x=$\frac{\sqrt{B_1}H}{\sqrt{B}-\sqrt{B_1}}$

Zadaci

1) Date su osnovna ivica $a = 10cm$ i visina $H = 12cm$ pravilne četvorostrane piramide. Odrediti njenu površinu i zapreminu.

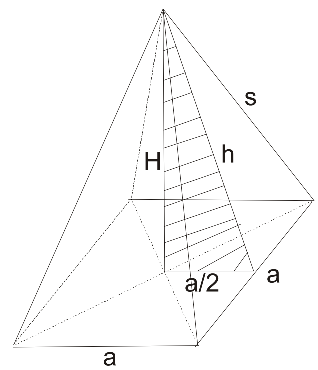
Rešenje:

$a = 10cm$

$H = 12cm$

$P = ?$

$V = ?$



Prvo ćemo naći visinu h:

$$h^2 = H^2 + \left(\frac{a}{2}\right)^2$$

$$h^2 = 12^2 + 5^2$$

$$h^2 = 169$$

$$\boxed{h = 13cm}$$

$$P = B + M$$

$$P = a^2 + 2ah$$

$$P = 10^2 + 2 \cdot 10 \cdot 13$$

$$P = 100 + 260$$

$$\boxed{P = 360cm^2}$$

$$V = \frac{BH}{3}$$

$$V = \frac{a^2 H}{3}$$

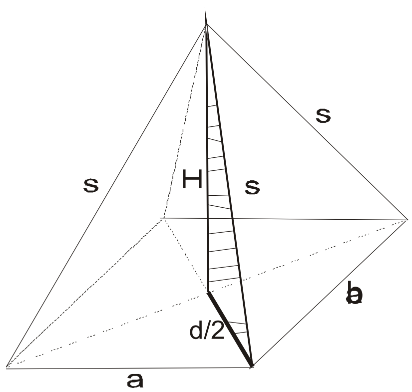
$$V = \frac{10^2 \cdot 12}{3}$$

$$V = 100 \cdot 4$$

$$\boxed{V = 400cm^3}$$

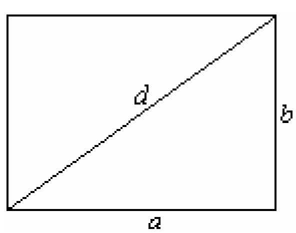
2) Osnova prave piramide je pravougaonik, sa stranicama 12cm i 9cm. Odrediti zapreminu piramide, ako je njena bočna ivica 12,5cm.

Rešenje:



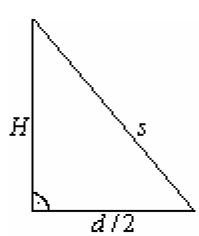
$$
\begin{aligned}
a &= 12cm \\
b &= 9cm \\
s &= 12{,}5cm \\
\hline
V &= ?
\end{aligned}
$$

Najpre nadjemo dijagonalu osnove (baze)



$$d^2 = a^2 + b^2$$
$$d^2 = 12^2 + 9^2$$
$$d^2 = 144 + 81$$
$$d^2 = 225$$
$$d = 15cm$$

Sada ćemo naći visinu H iz trougla.



$$H^2 = s^2 - \left(\frac{d}{2}\right)^2$$
$$H^2 = 12{,}5^2 - 7{,}5^2$$
$$H^2 = 100$$
$$H = 10cm$$

$$V = \frac{1}{3}BH$$
$$V = \frac{1}{3}abH$$
$$V = \frac{1}{3}12 \cdot 9 \cdot 10$$
$$V = 360cm^2$$

3) Osnova prizme je trougao čije su stranice 13cm, 14cm i 15cm. Bočna ivica naspram srednje po veličini osnovne ivice normalna je na ravan osnove i jednaka je 16cm. Izračunati površinu i zapreminu piramide.

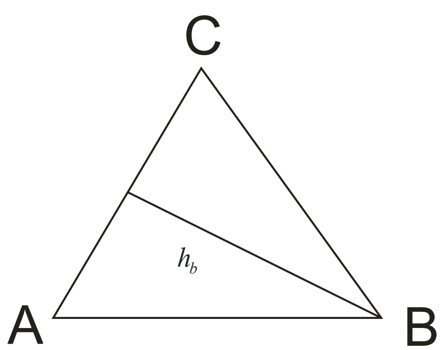
Rešenje:

Nadjimo najpre površinu baze preko Heronovog obrasca.

$$a = 13cm \quad b = 14cm \quad c = 15cm \quad \Rightarrow \quad s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$$B = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 7 \cdot 8 \cdot 6} = 84cm^2$$

Nama treba dužina srednje po veličini visine (h_b) osnove.

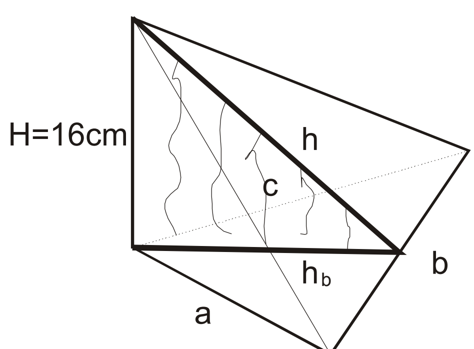


$$P = \frac{b \cdot h_b}{2} \Rightarrow \quad 84 = \frac{14 \cdot h_b}{2}$$

$$84 = 7h_b$$

$$h_b = 12cm$$

Naći ćemo dalje visinu bočne strane h.



$$h^2 = H^2 + h_b$$

$$h^2 = 16^2 + 12^2$$

$$h^2 = 256 + 144$$

$$h^2 = 400$$

$$h = 20cm$$

Površina piramide je jednaka zbiru površina ova četiri trougla!

$$P = B + \frac{a \cdot H}{2} + \frac{c \cdot H}{2} + \frac{bh}{2}$$

$$P = 84 + \frac{13 \cdot 16}{2} + \frac{15 \cdot 16}{2} + \frac{14 \cdot 20}{2}$$

$$P = 84 + 104 + 120 + 140$$

$$P = 448cm^2$$

$$V = \frac{1}{3}BH$$

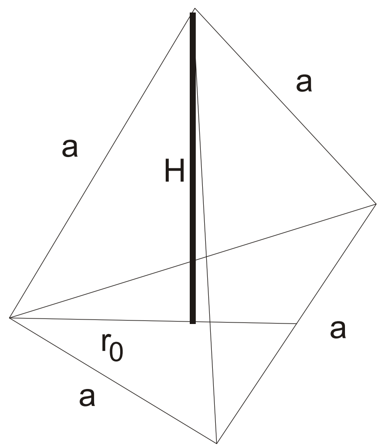
$$V = \frac{1}{3}84 \cdot 16$$

$$V = 448cm^3$$

4) Izračunati zapreminu pravilnog tetraedra u funkciji ivice a

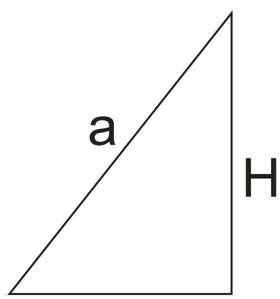
Rešenje:

Tetraedar je pravilna jednakoivična trostrana piramida.



$$V = \frac{1}{3} BH$$

Izvucimo trougao:



$$r_o = \frac{a\sqrt{3}}{3}$$

$$H^2 = a^2 - \left(\frac{a\sqrt{3}}{3}\right)^2 = a^2 - \frac{a^2 \cdot 3}{9} = \frac{9a^2 - 3a^2}{9} = \frac{6a^2}{9}$$

Dakle:

$$H = \frac{a\sqrt{6}}{3}$$

$$V = \frac{1}{3} \frac{a^2\sqrt{3}}{4} \cdot \frac{a\sqrt{6}}{3}$$

$$V = \frac{a^3\sqrt{18}}{36}$$

$$V = \frac{a^3 \cdot 3\sqrt{2}}{36}$$

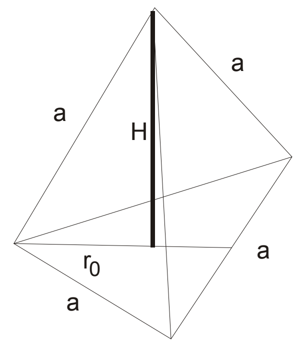
PAZI: $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$

$$V = \frac{a^3 \cdot \sqrt{2}}{12}$$

5) Izraziti visinu pravilnog tetraedra u funkciji zapremine V.

Rešenje:

Iskoristićemo rezultat prethodnog zadatka



$$V=\frac{a^3\sqrt{2}}{12}$$ i **izraziti** a

$$a^3=\frac{12V}{\sqrt{2}}$$

$$a^3=\frac{12V}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$a^3=6\sqrt{2}V$$

$$a=\sqrt[3]{6\sqrt{2}V}$$

$$a=\sqrt[3]{6}\sqrt[6]{2}\sqrt[3]{V}$$

Kako je

$H=\frac{a\sqrt{6}}{3}$ to je

$$H=\frac{\sqrt[3]{6}\sqrt[6]{2}\sqrt[3]{V}\sqrt{6}}{3}$$

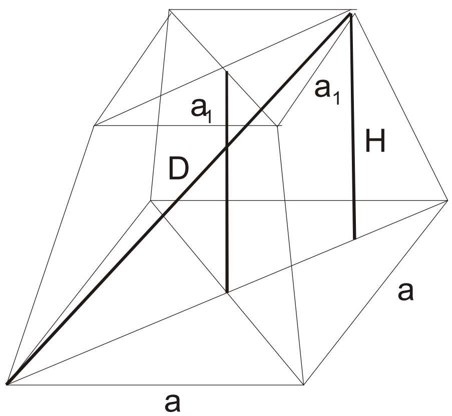
$$H=\frac{\sqrt[6]{6^2}\cdot\sqrt[6]{6^3}\cdot\sqrt[6]{2}\cdot\sqrt[3]{V}}{3}$$

$$H=\frac{\sqrt[6]{6^5\cdot 2}\sqrt[3]{V}}{3}=\frac{\sqrt[6]{2^5\cdot 3^5\cdot 2}\sqrt[3]{V}}{3}$$

$$\boxed{H=\frac{2\sqrt[6]{3^5}\sqrt[3]{V}}{3}}$$

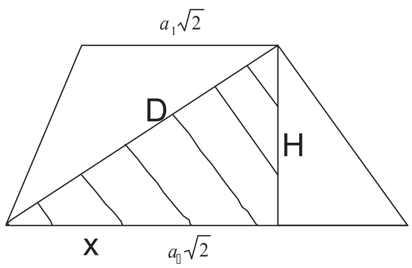
6) Izračunati zapreminu pravilne četvorostrane zarubljene piramide ako su osnovne ivice 7m i 5m i dijagonala 9m.

Rešenje:



$$a = 7m$$
$$a_1 = 5m$$
$$D = 9m$$
$$\overline{\qquad\qquad}$$
$$V = ?$$

Da bi našli visinu H moramo uočiti dijagonalni presek.



$$x = \frac{a\sqrt{2} + a_1\sqrt{2}}{2}$$
$$x = \frac{7\sqrt{2} + 5\sqrt{2}}{2}$$
$$x = 6\sqrt{2}m$$

$$D^2 = H^2 + x^2$$
$$H^2 = D^2 - x^2$$
$$H^2 = 9^2 - \left(6\sqrt{2}\right)^2$$
$$H^2 = 81 - 72$$
$$H^2 = 9$$
$$H = 3m$$

$$V = \frac{H}{3}\left(B + B_1 + \sqrt{BB_1}\right)$$
$$V = \frac{H}{3}\left(a^2 + a_1^2 + aa_1\right)$$
$$V = \frac{3}{3}\left(7^2 + 5^2 + 7 \cdot 5\right)$$
$$V = 109m^3$$

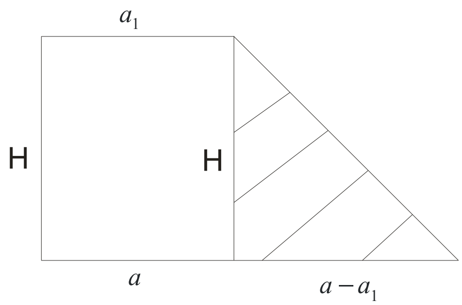
7) Izračunati zapreminu pravilne šestostrane zarubljene piramide ako su osnovne ivice 2m i 1m i bočna ivica 2m

Rešenje:

$a = 2m$

$a_1 = 1m$

$s = 2m$



$H^2 = s^2 - (a - a_1)^2$

$H^2 = 2^2 - 1^2$

$H^2 = 3$

$H = \sqrt{3}$

$$V = \frac{H}{3}\left(B + B_1 + \sqrt{BB_1}\right)$$

$$V = \frac{H}{3}\left(\frac{6a^2\sqrt{3}}{4} + \frac{6a_1^2\sqrt{3}}{4} + \frac{6aa_1\sqrt{3}}{4}\right)$$

$$V = \frac{\sqrt{3}}{3} \cdot \frac{6\sqrt{3}}{4}\left(2^2 + 1^2 + 2 \cdot 1\right)$$

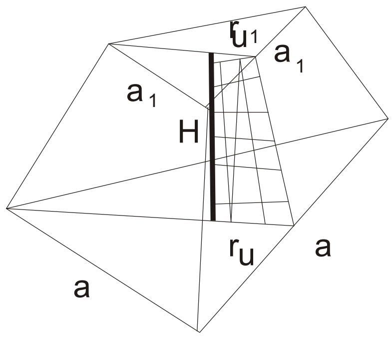
$$V = \frac{3}{2} \cdot 7$$

$$V = \frac{21}{2}$$

$$V = 10,5m^3$$

8) Osnovne ivice pravilne trostrane zarubljene piramide su 2cm i 6cm. Bočna strana nagnuta je prema većoj osnovi pod uglom od 60^o. Izračunati zapreminu te piramide.

Rešenje:

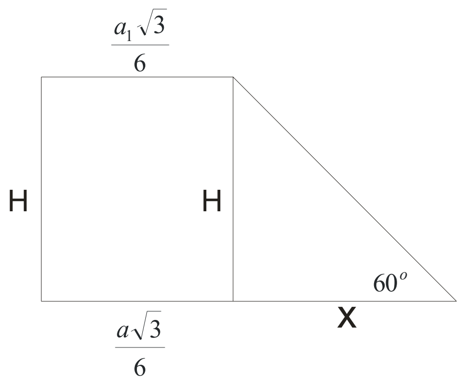


$a = 6cm$

$a_1 = 2cm$

PAZI: Kad se u zadatku kaže bočna strana pod nekim uglom, to je ugao izmedju visina bočne strane i visine osnove!

Izvucimo "na stranu" trapez (pravougli)



$$x = \frac{a\sqrt{3}}{6} - \frac{a_1\sqrt{3}}{6} = \frac{6\sqrt{3}}{6} - \frac{2\sqrt{3}}{6} = \frac{4\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$$

$$tg60^o = \frac{H}{x} \Rightarrow H = x \cdot tg60^o = \frac{2\sqrt{3}}{3} \cdot \sqrt{3} = 2cm$$

$$V = \frac{2}{3}\frac{\sqrt{3}}{4}\left(6^2 + 2^2 + 6 \cdot 2\right)$$

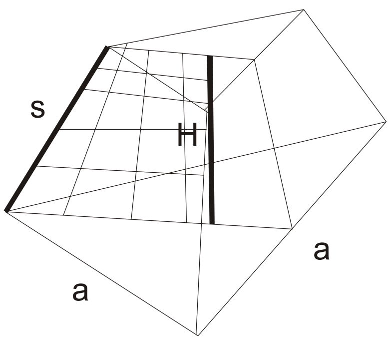
$$V = \frac{\sqrt{3}}{6}\left(36 + 4 + 12\right)$$

$$V = \frac{\sqrt{3}}{6} \cdot 52$$

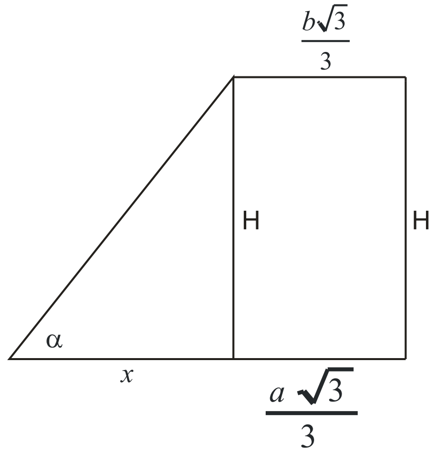
$$V = \frac{26\sqrt{3}}{3} m^3$$

9) Bočne ivice pravilne trostrane zarubljene piramide nagnute su prema ravni osnove pod uglom α. Osnovne ivice piramide su a i b $(a > b)$. Odrediti zapreminu piramide.

Rešenje:



Izvucimo obeleženi trapez, iz njega ćemo naći visinu!



$$x = \frac{a\sqrt{3}}{3} - \frac{b\sqrt{3}}{3} = \frac{(a-b)\sqrt{3}}{3}$$

$$tg\alpha = \frac{H}{x}$$

$$\Downarrow$$

$$H = xtg\alpha = \frac{(a-b)\sqrt{3}}{3} \cdot tg\alpha$$

$$V = \frac{H}{3}\left(\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \frac{ab\sqrt{3}}{4}\right)$$

$$V = \frac{1}{3}\frac{(a-b)\sqrt{3}}{3} \cdot tg\alpha \cdot \frac{\sqrt{3}}{4}(a^2 + b^2 + ab)$$

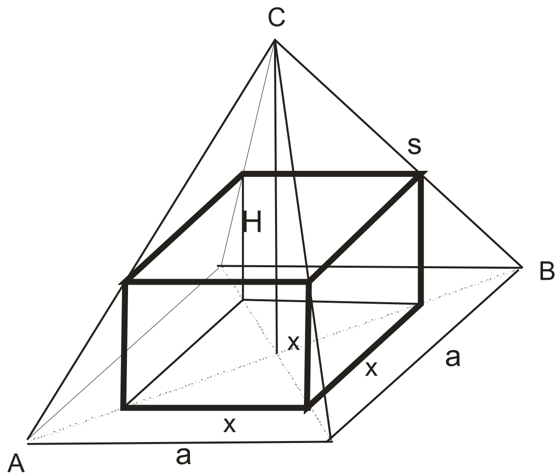
$$V = \frac{(a-b)tg\alpha}{12}(a^2 + b^2 + ab)$$

Kako je $(a-b)(a^2 + b^2 + ab) = a^3 - b^3$

$$\boxed{V = \frac{(a^3 - b^3)tg\alpha}{12}}$$

10) Data je prava pravilna četvorostrana piramida osnovne ivice $a = 5\sqrt{2}cm$ i bočne ivice s=13cm. Izračunati ivicu kocke koja je upisana u tu piramidu tako da se njena četiri gornja temena nalaze na bočnim ivicama piramide.

Rešenje:



$a = 5\sqrt{2}cm$

$s = 13cm$

Nadjimo najpre visinu piramide.

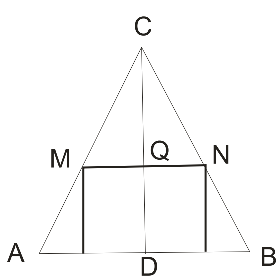
$$H^2 = s^2 - \left(\frac{a\sqrt{2}}{2}\right)^2$$

$$H^2 = 13^2 - \left(\frac{5\sqrt{2}\sqrt{2}}{2}\right)^2$$

$$H^2 = 144$$

$$H = 12cm$$

Izvucimo "na stranu'' dijagonalni presek:



Dobili smo 2 slična trougla: $\Delta ABC \sim \Delta MNC$

PAZI:

→ AB je dijagonalna osnove $AB = a\sqrt{2} = 5\sqrt{2}\sqrt{2} = 10cm$

→ MN je dijagonala stranice kvadrata $MN = x\sqrt{2}$

→ Visina CD=H=12cm

→ Visina CQ=H-x=12-x

Dakle:

$$AB : MN = CD : CQ$$

$$10 : x\sqrt{2} = 12 : (12 - x)$$

$$10(12 - x) = 12 \cdot x\sqrt{2}$$

$$120 - 10x = 12\sqrt{2}x$$

$12\sqrt{2}x + 10x = 120 \rightarrow$ Podelimo sa 2

$$6\sqrt{2}x + 5x = 60$$

$$x(6\sqrt{2} + 5) = 60$$

$x = \dfrac{60}{6\sqrt{2} + 5} \rightarrow$ Racionališemo

$$x=\frac{60}{6\sqrt{2}+5}\cdot\frac{6\sqrt{2}-5}{6\sqrt{2}-5}$$

$$x=\frac{60(6\sqrt{2}+5)}{72-25}$$

$$\boxed{x=\frac{60(6\sqrt{2}+5)}{47}}$$

Ovo je tražena ivica kocke.

11) Osnova piramide je tangentni poligon sa n stranica opisan oko kruga poluprečnika r. Obim poligona je 2p, bočne stranice piramide nagnute su prema ravni osnovne pod uglom φ. Odrediti zapreminu piramide.

Rešenje:

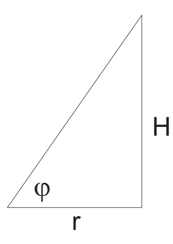
Baza ove piramide je sastavljena iz n-trouglova. Ako stranice poligona obeležimo sa $a_1, a_2....a_n$, onda će površina svakog od tih n-trouglova biti $P_i=\frac{a_i\cdot r}{2}$, odnosno

$$B=P_1+P_2+...P_n$$

$$B=\frac{a_1r}{2}+\frac{a_2r}{2}+...+\frac{a_nr}{2}=\frac{r}{2}(a_1+a_2+...a_n)\rightarrow \text{ gde je } a_1+a_2+...a_n \text{ obim poligona}$$

$$B=\frac{r}{2}\cdot 2p=rp$$

Pošto kaže da su bočne stranice nagnute pod uglom φ, to je:



$$tg\varphi=\frac{H}{r}\Rightarrow H=rtg\varphi$$

$$V=\frac{1}{3}BH$$

$$V=\frac{1}{3}rp\cdot rtg\varphi$$

$$\boxed{V=\frac{r^2p\cdot tg\varphi}{3}}$$